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FORMULATION OF ELASTIC LOADING PARAMETERS FOR STUDIES OF CLOSED-CIRCUIT UNDERWATER BREATHING SYSTEMS

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Peak-to-peak mouth or oro-nasal pressure is used as an indicator of flow resistance during unmanned testing of Underwater Breathing Apparatus (UBA). This interpretation can be confounded by UBA elastance. In modern UBA's elastic pressures can be greater than resistive pressures at low flow rates, resulting in a miscalculation of the work of breathing. Not only must elastance be properly accounted for in UBA testing, but an understanding of elastic loads in general must be acquired. To the latter end, this work provides a rigorous description of elastic loads encountered in the use of breathing systems underwater. Elastance equations were derived from first principles, and compared with experimental measurements. The pressure-volume characteristics of boxes, cylinders, pyramids, and various breathing bags were examined, as was a U.S. Navy MK-15 closed-circuit UBA. The major determinant of elastance underwater was the geometry of the deformable gas volume, with elastic pressures generated by the movement of air-water interfaces. Both the behavior of various elastances, and the errors involved in common

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assumptions about elastance were detailed. For simple geometries, theory and measurement agreed closely. In spite of the complex geometry of the MK-15 breathing bag, the MK-15 elastance approximated that of a cylindrical water column. A submerged MK-15 oriented vertically and horizontally, yielded elastances of 4 and 1.8 cm H2O/1, respectively.

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NOMENCLATURE

A = cross sectional area, cm^2

B = length of base side, square-base pyramid, cm

C = compliance, Vt/h, liters/cm H_2O

C1 = a constant including material properties of the rubber

D = cone base diameter or cylinder diameter, cm

e = base of natural logarithms, 2.7183

E' = elastic modulus, Young's modulus, dynes/cm^2

E = elastance, elastic load, h/Vt, cm H_2O/liter

h = pressure over atmospheric or ambient, gauge pressure, cm H_2O

hi = initial air space height for water column container, cm

H = height of cone or pyramid, base to apex, cm

m = slope in linear equation

P = pressure, cm H_2O, etc.

ΔP = pressure difference, cm H_2O, etc.

q = intercept in linear equation

r = radial coordinate

R = radius of cone base or cylinder, cm

T = tensile stress, dynes/cm^2

V = volume, liters, etc.

V* = volume of material in rubber balloon, cm^3

Vb = balloon volume, cm^3 or liters

Vbox = volume of non-collapsible enclosed space, liters

Vci = initial volume of air in water column container, liters

Vh = volume due to water column depression or movement from initial condition, liters

V0 = volume at which spherical shape of balloon first achieved, liters

Vsi = initial volume of syringe, liters

Vsumi = Vx + Vsi + Vci

Vt = tidal volume, liters

Vx = volume of tubing, fittings, etc., liters

x = distance coordinate

z = axial distance coordinate

zc = axial length of cone or pyramid cap, apex to frustum plane, cm

Greek Symbols

δ = thickness, cm

ε = strain, dimensionless

ϕ = cone or pyramid half-angle at apex

θ = angular coordinate

1.0 INTRODUCTION

To use closed-circuit underwater breathing apparatus (UBA) a diver must overcome a variety of impedances; those imposed by hydrostatic imbalances, by resistance, by inertance and by the subject of this report, elastance. Flow resistance is created by uni-directional valves, breathing hoses, and CO_2 canisters. Hydrostatic loads are generated by differences in placement of lungs and breathing bags. The motion of air within breathing bags is a primary generator of the third type of impedance, elastance. Like the first two impedances, elastance may impair diver comfort and exercise tolerance, and may contribute to the work of breathing (1,2). Elastance is also the least understood impedance, both in terms of its physiological effect on divers and of its dependence on UBA design.

During unmanned testing of UBA's, peak-to-peak pressure swings (ΔP) measured at the mouthpiece or within an oro-nasal mask serve as an index of flow impedance. The ΔP's are generated by elastance, resistance and inertance, so interpretating peak pressure is not simple.

For a given ΔP, elastance does not seem to produce the same subjective sensations as does resistance. For example, in recent unmanned tests of two prototype closed-circuit UBA at the Navy Experimental Diving Unit (NEDU), one rig had higher ΔP's than the other. By the usual convention, the rig with the lower pressure would be considered a better UBA. However, during manned testing, the rig with higher ΔP was preferred. While differences in hydrostatic loading may have contributed to the divers' subjective sensations, it was also observed that the UBA with higher ΔP had a lower flow resistance. The higher ΔP of this UBA was apparently due to a relatively high but seemingly innocuous elastance.

At the Naval Medical Research Institute (NMRI) divers compared the better of the above prototypes with the operational MK-15. Again, the prototype was preferred by the divers, in spite of a higher ΔP generated by the UBA's elastance. Obviously, the interaction between resistance and elastance is playing a significant but poorly understood role in diver acceptance of new UBA.

The primary determinant of elastance in closed-circuit UBA is the shape of breathing bags and the manner in which they fill with gas. To date, there has been no thorough and formal explanation of the influence of bag shape and orientation on UBA elastance. This report is the first effort in that direction.

2.0 DEFINITIONS

Elastance (E) is the reciprocal of compliance (C). Elastance expresses the extent to which pressure changes in an enclosed system for a change in system volume. The units of elastance are commonly cm H_2O/liter.

The equation of motion for a mechanical system where the major variables are pressure and volume is:

$$\Delta P = (1/C)\ V + R\dot{V} + I\ddot{V} \qquad (1)$$

where ΔP is the pressure difference applied across the system, C is system compliance, R is resistance, and I is inertance. V represents volume; $\dot{V}$ (the the first derivative of V with respect to time) represents flow, and $\ddot{V}$ (the second derivative) represents acceleration. Substituting E for 1/C, equation (1) becomes:

$$\Delta P = EV + RV + IV \qquad (2)$$

When the system of interest is linear, E, R, and I are constant. To a first approximation, the human respiratory system is linear, however, we will not restrict our discussions to linear systems, either biological or man-made. We should thus consider a generalization of equation (2):

$$\Delta P = f1(E) + f2(R) + f3(I) \qquad (3)$$

where nonlinearities of E, R, and I are contained within the functions f1, f2, and f3, respectively. Due to the ambiguity inherent within the three functions, the conditions of volume, flow, and acceleration must be given for any measure of ΔP. All of the following elastance measurements were made under zero flow and zero acceleration conditions, therefore resistance and inertance were zero.

The typical definition of an elastic material is one that returns to its original shape after being subjected to a deforming stress. We define not an elastic material or substance, but an elastic system. Such a system impedes a change of volume within it. Volume changes in elastic systems can only occur through the expenditure of energy, thereby changing system pressure.

System elastance may be contributed by elastic material, e.g. rubber neck dams or breathing bags, by gas compressibility, or by the hydrostatic pressure gradients involved in moving air in water columns. The system in which elastance acts may be rigid (a box), totally flexible (a breathing bag or balloon), or a partly deformable enclosure (a water column).

This investigation into the fundamental aspects of elastic loading includes various elastic loading geometries, such as a) a rigid container or box, b) a water column with both linear (vertical cylinder) and nonlinear (conical/pyramidal) characteristics, c) various elastic balloons including breathing bags, and finally d) an operational US Navy closed circuit UBA (MK-15). Equations describing the pressure-volume relationships for the above geometries were derived from first principles, then tested experimentally. The results should serve as a reference for future investigations into the physiological effects of elastic loading, and aid in the interpretation of elastance measurements of underwater breathing apparatus.

2.1 Previous Literature

The scant literature on the physiological effects of elastic loading deals either with rigid containers, where elastance is contributed by the compressibility of the enclosed gas, or with the elastic properties of the lungs. Most existing work (3) deals with the physiological aspects of elastic loading, i.e. the psychophysical threshold for awareness of elastance, the tolerance to elastance, the relation of elastic load to peak mouth pressure, and the effect of elastance on exercise performance.

To our knowledge the quantitative aspects of elastance of deformable spaces have not been described. A general fundamental look at the non-collapsible enclosed volume (the box) has also been lacking.

3.0 THEORY

In the following analyses, the elastic load is considered in conjunction with a volume changing element (syringe, lung, breathing machine). In this way a closed system is formed; the total moles of gas are always constant, and if isothermal conditions and unity compressibility factor are assumed, Boyle's Law can be rigorously applied. However, in the resulting system, both pressure and system volume (total volume) will change in a manner determined by the overall system elastance.

3.1 The Box

We begin with the box, not because it is found in UBA, but because it is a relatively simple way to introduce the subject. The box is furthermore the classical way of producing pure elastic respiratory loads.

A box is a rigid enclosed volume to which Boyle's Law, $P_1V_1 = P_2V_2$, may be applied. Mass must be added to change the pressure inside the box, since the volume of the enclosure is always constant. This can be accomplished by transferring gas from a non-rigid component of the total system. In a box, the total volume consists of the box volume, the initial volume of the volume adding element such as a syringe, and the volume of the tubing connecting the syringe to the box. Thus,

$$Vi = Vbox + Vsi + Vx \tag{4}$$

where,

Vi = initial (total) volume

Vbox = box volume (a constant)

Vsi = initial syringe volume, and

Vx = volume of connecting tubing (another constant).

At the final conditions,

$$Vf = Vbox + Vsf + Vx \tag{5}$$

where,

Vf = final (total) volume

Vsf = final syringe volume

But,

$$Vsf = Vsi - Vt \tag{6}$$

where,

Vt = tidal volume or volume moved by syringe

By convention the volume transferred from the syringe is tidal volume, and has a positive sign.

Assuming atmospheric pressure for the initial conditions, Boyle's Law can be written,

$$Pi(Vi) = Pf(Vf) \tag{7}$$

Where Pi = initial pressure (assumed ambient) and Pf = final pressure.

Rearranging,

$$\frac{Pf}{Pi} = \frac{Vsi + Vx + Vbox}{Vsi + Vx + Vbox - Vt} \tag{8}$$

But $Pf = Pi + h$, where h is the gauge pressure. Substituting and rearranging,

$$h = \frac{Pi \cdot Vt/(Vsi + Vx + Vbox)}{1 - Vt/(Vsi + Vx + Vbox)} \qquad (9)$$

This is the (gauge) pressure-volume relationship for the box, where pressure is in cm H_2O and volume in liters. The equation is nonlinear and very general. For those physiological studies (4,5,6) where a box was used to generate a known elastic load, Vt was much smaller than the box volume, and peak pressures relative to atmospheric were low. Therefore, a simplification can be made by ignoring the Vt term in the denominator. This yields,

$$h = [Pi/(Vsi + Vx + V_{box})]\ Vt = E \cdot Vt \qquad (10)$$

which is a linear relationship. The quantity in brackets is the elastance (E).

If the box volume is much greater than the other volumes in the sum, then

$$h = [Pi/Vbox]\ Vt \qquad (11)$$

which is the form used in the physiological literature. The h term may be interpreted as pressure over ambient, which need not be atmospheric. Pi is the initial (absolute) pressure in the box.

3.2 Water Column

A movable air-water interface comprises an elastic load since a volume change displacing the interface downward requires an increase in system

pressure. Here we describe the elastance of an open-ended enclosure; the upper end allowing gas to enter or leave, and the lower end permitting the transfer of water. The shape of the enclosure is arbitrary. To minimize computational difficulties and yet allow both linear and nonlinear elastance, five geometrical configurations were studied. Those included a vertical cylinder, a vertical cone - apex up and down, and a vertical pyramid - apex up and down. The cone was not tested experimentally because it could not be easily fabricated. The analysis for both cone and pyramid are found in Appendix B. They follow closely, except for more complicated algebra, the development of the cylinder given below.

The treatment of system volumes in the vertical cylinder is the same as for the box, except that the volume of the enclosure is variable as a result of the water column. Fig. 1 shows a typical experimental arrangement.

The cylinder is placed vertically, so that the air-water interface area corresponds to the cylinder cross-sectional area (A). The system may contain an initial cylinder volume, (Vci) depending on the height of the cylinder top above the water. The difference in pressure between gas inside and outside the cylinder results in, and is measured by, the difference in water levels. The volume of water displaced from initial condition is equal to A•h.

Thus,

$$\frac{Pf}{Pi} = \frac{Vci + Vx + Vsi}{Vci + Vx + Vsi - Vt + A \cdot h} \qquad (12)$$

Converting to gauge pressure h, as before, and cross multiplying gives,

$$Pi\ (Vci + Vx + Vsi) = (h + Pi)\ (Vci + Vx + Vsi - Vt + A \cdot h) \qquad (13)$$

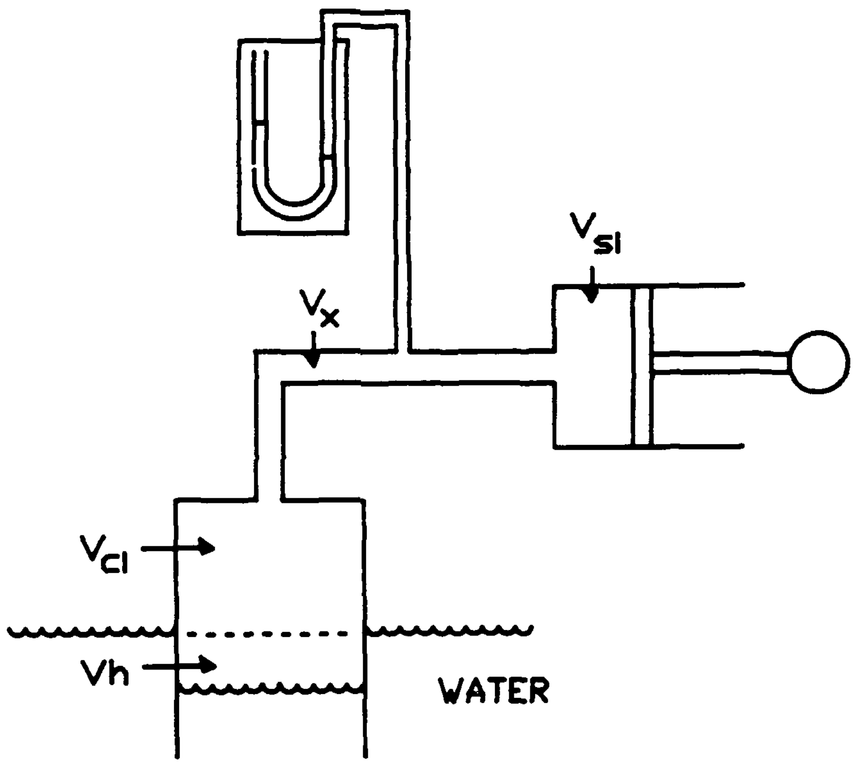


Fig. 1. The Vertical-Cylinder Water Column

which is a quadratic equation in h. Multiplying out the terms, canceling and rearranging into quadratic form gives,

$$h^2 + [Pi + (Vsumi - Vt)/A]\ h - Pi(Vt/A) = 0 \tag{14}$$

where Vsumi = Vci + Vx + Vsi. The typical units of volume are liters, and normally the units of A•h are cm^3. Therefore, a units conversion (1 liter = 1000 cm^3) will need to be applied. When A is in cm^2 and h is in cm, A•h/1000 will be in liters, and the units in equation 12 will be consistent. The quadratic formula can then be used to solve for h in terms of Vt. Here a = 1, b = Pi + (Vsumi - Vt)(1000/A), and c = - Pi (1000Vt/A), where a, b and c are the coefficients, respectively, of the powers of h starting with the highest. Thus, since a = 1,

$$h = \pm\ b/2 + .5 \sqrt{(b^2 - 4\ c)} \tag{15}$$

Only the positive root of the quadratic solution has physical meaning.

Equation 12 is general but nonlinear. Simplification can be made by an order of magnitude analysis of the terms in equation 14. The h^2 term is likely to be small relative to Pi•h when Pi is atmospheric (i.e. 1033 cm H_2O), therefore it can be neglected. This gives,

$$(Pi \cdot A + Vsumi - Vt)\ h = Pi(Vt) \tag{16}$$

Since 1033 Vt is much greater than h•Vt, the h•Vt term can be neglected.

Using the units conversion for A, the pressure - volume relationship for the water column can be written,

$$h = \frac{1}{(A/1000 + Vsumi/Pi)} Vt \qquad (17)$$

The elastance of the water column is then the term multiplying Vt.

A further simplification can be made by assuming Vsumi/Pi is much smaller than A/1000. Then,

$$h = (1000/A)\ Vt \qquad (18)$$

and the elastance is simply 1000/A with A in cm^2. This form is useful as a first approximation, but may give errors of 20% or more at large Vt or with small diameter cylinders.

In Table 1 both the quadratic and simple solution for gauge pressure and elastance for various combinations of cylinder size and initial syringe (breathing machine or lung) volumes are tabulated. The initial cylinder volume was assumed to be 1 liter (V_{ci} = 1), which is a reasonable value for initial gas volume in a UBA breathing bag at the beginning of expiration.

TABLE 1
Water Column Elastance - Measurements and Comparison with Models (Vci = 1, and V_x = 0)).

		Quadratic Model		Simplified Model	
A	Vsi	hquad	Equad	hsim	Esim
40	6	43.0	21.5	50	25
40	2	46.7	23.4	50	25
67	6	27.2	13.6	30	15
67	3	28.3	14.2	30	15
120	6	15.8	7.9	16.7	8.3

A = cylinder cross-section in cm^2, Vsi = initial syringe volume in liters, hquad = quadratic solution (Eqn. 15), hsim = simplified solution (Eqn. 18). P_i = atmospheric, V_t = 2 liters

In general, the greater the initial lung volume the greater the discrepancy between the simple and complete solution. The smaller the cylinder size, the greater its elastance, and again the greater the difference between predictions. In all of these examples, the simplest solution overestimated the measured elastance.

3.3 Elastic Bag

Although it is commonly assumed that the elasticity of breathing bag materials is the major source of elastance in closed-circuit UBA, that is not really the case. As demonstrated later, the water column effect seems to be more important, at least at mid-bag volumes. For completeness, however, we include the following analysis of the pressure-volume characteristics of a spherical balloon of purely elastic (rubber) material in air. Details are

provided in Appendix C. The form of the governing equation is too complex to permit a single solution of Vt as a function of h. Therefore, the complex P-V relationship for a balloon, illustrated in Fig. 2, was broken into three linearized regions.

In Zone I the balloon is flaccid, like a UBA breathing bag far removed from either the fully distended or fully collapsed state. In zone I, h = 0 for all Vt, and elastance is zero.

Zone II is the region of sharply increasing pressure due to the elastic resistance of the balloon material. This zone is modeled by a straight line,

$$h = m V + q \tag{19}$$

where m is the slope and q is the y- intercept of the line of best fit for the data between V0 and Vpmax. The y- intercept will be negative in all cases.

Zone III is a region of negative elastance (the balloon is "easier" to inflate once past the point of maximum pressure). This zone is modeled by an equation whose form is identical to that above, except that the slope is negative and the y- intercept is positive (m and q must be determined experimentally in both zones II and III). The explanation for a pressure maximum is shown in Appendix C. The location of the inflection point is also developed there.

The general analysis with the linearized elastance gives a quadratic in h, which is solved the same way as before (Appendix C). The simplified version obtained by order of magnitude analysis can be used as before, but obviously, the dominant effect is due to the balloon. Therefore, the simplified equations describing the elastance of the balloon are the same as

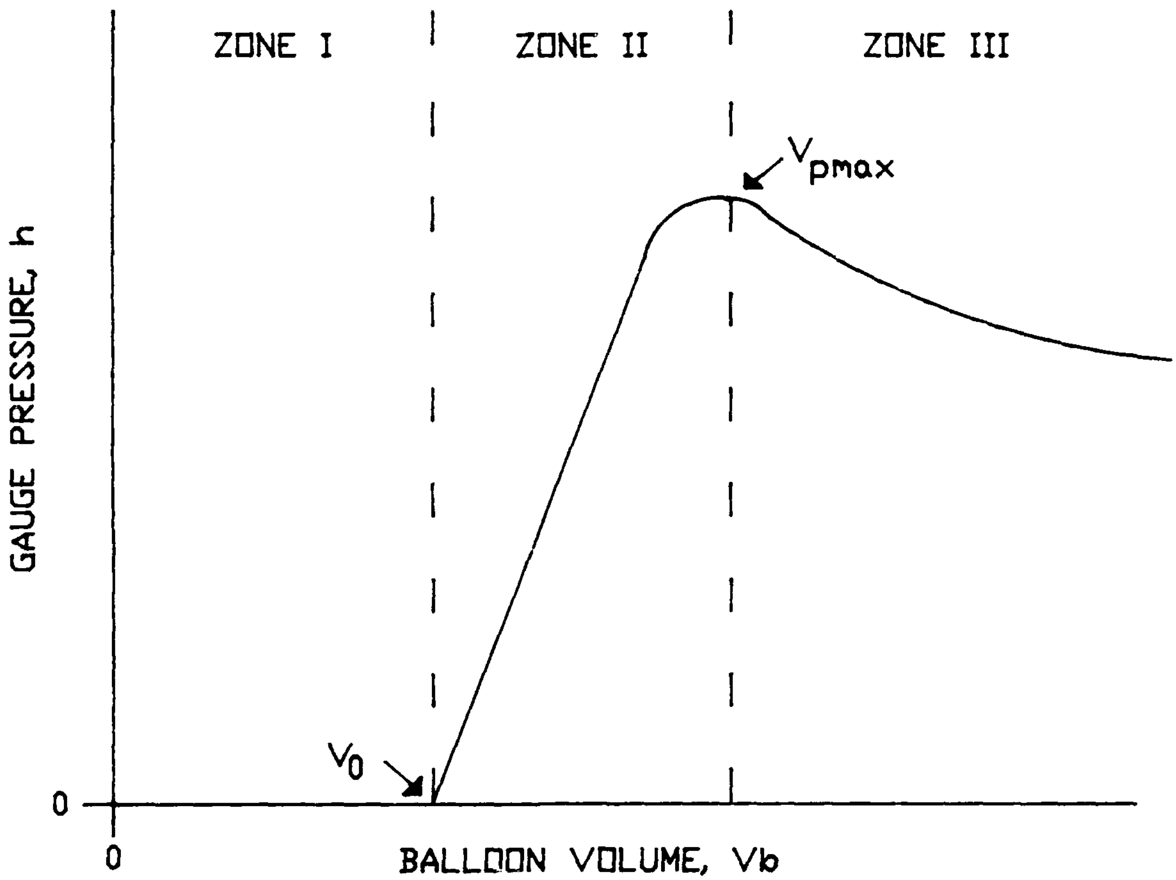


Fig. 2. Three-Zone Division of the Balloon Pressure-Volume Curve

those for zones II and III given previously. The elastance in both cases must be determined by experiment. This was done on various balloons and breathing bags; these results and those for other geometries are given in the next section.

4.0 EXPERIMENTAL TEST OF ELASTIC LOADING MODELS

4.1 The Box

A 3-liter calibration syringe was connected to a 88.5 l Plexiglas box by a short length of relatively non-elastic tubing. A water manometer, connected to the system through a pressure tap, measured the pressure in the box, relative to atmospheric (h = gauge pressure in cm H_2O). The volume of the connecting tubing (Vx) was 0.15 liters which was negligibly small. The measured or experimental values of h (hexp) and the values predicted from the linear (hlin, eqn. 10) and simplified (hsim, eqn. 11) theoretical models are compared in Table 2.

TABLE 2

Elastic Load Prediction and Comparison with Data for a Box (Vbox = 88.5 liters)

Vt	hexp	hlin	hsim
1	12	11.4	11.7
2	23	23.1	23.3
3	35	35.0	35.0

where Vt is in liters, and h in cm H_2O.

In spite of imprecision in the measurement, the agreement between theory and practice is excellent. The elastance calculated by the simplified model was 11.67 cm $H_2O/1$.

4.2 Water Column

Experimental data for a water column elastic load were obtained using a rigid cylinder, approximately 80 cm long and 19.5 cm in diameter. The cylinder was placed vertically in a water bath, with its top (closed) part 20 cm above the water surface. This yielded an initial air space volume (Vci) of 5.97 liters. The pressure - volume data were then compared to theoretical models. Table 3 gives the results.

TABLE 3

Water Column Elastance Measurements and Comparison with Models - 20 cm Free Space (Vci ≅ 6 liters)

Vsi	Vsf	Vt	hexp	hquad	hs17	hs18
3	0	3	9.7	9.6	9.7	10.45
2	0	2	6.5	6.4	6.48	6.7
1	0	1	3.2	3.2	3.25	3.35
3	1	2	6.5	6.5	6.46	6.7
3	2	1	3.2	3.2	3.23	3.35

All volumes (V) are given in liters and pressures (h) in cm H_2O.

The pressure measured experimentally is hexp; hquad is the prediction from the quadratic model (eqn. 15), hs17 represents the simplified model based upon eqn. 17, and hs18 is from the highly simplified eqn. 18. The tidal volume (Vt) is the difference between initial and final volume in the system. Elastance was approximately 3.3 cm H_2O/l.

The agreement between the first two models and the data was excellent. The highly simplified model showed good agreement also except at high tidal volume. The deviation is accentuated by an increase in elastance (see Fig 14, Appendix A-2). On the other hand, equation 17 is quite adequate for predicting system pressures over a wide range of elastances.

4.2.1 Testing Machine Independence

During unmanned testing of UBA, a diver's ventilation is simulated by a breathing machine; essentially a syringe connected to the UBA through lengths of tubing. For closed-circuit UBA with breathing bags, the UBA elastance may be similar to that of the vertical cylinder, so the preceding discussion applies.

Elastance of such UBA can be determined by static P-V curves. However, as demonstrated here, the results of such tests (elastance values) will depend upon the testing system, i.e. volume contained within connecting tubing, initial volume within the breathing machine, and even the end-inspiratory volume within the UBA breathing bags. In other words, measured elastance will be test specific, unless these factors are taken into account.

As an example, suppose three different testing laboratories, each with slightly different test conditions, were assigned to determine the elastance of a UBA. Hypothetical elastances are calculated using E = h/Vt in each case, where tidal volume is the same but the initial syringe volume (Vsi) and tubing volume (Vx) vary. The results are shown in Table 4.

TABLE 4

Test Dependent Elastance

Vx	Vsi	Vt	h	Etd
0.2	2	2	47.58	23.79
2	6	2	41.34	20.67
1	6	2	43.01	21.50

h is system (gauge) pressure in cm H_2O measured under static conditions, and Etd is the test dependent elastance in cm H_2O/liter.

In spite of the fact that the tidal volume was the same at all three laboratories, three different estimates for system pressure and elastance were obtained. Thus, in order to accurately measure elastance, and to have valid comparisons between laboratories, the tubing volume and initial syringe volume, as well as the tidal volume, must be known.

Obviously, a test-independent elastance of a UBA is desired. Since the UBA elastance can be modeled by a water column effect, the elastance can be determined in the following manner. Rearranging equation 16 yields,

$$A = (Pi \cdot Vsumi/(h + Pi) - Vsumi + Vt)/h \qquad (20)$$

where again, Vsumi = Vci + Vx + Vsi. With volume in liters and pressure in cm H_2O. The A term needs the conversion factor, 1000 cm^3/liter.

In most testing situations, both A and Vci (the initial volume of gas in the breathing bag) may be unknown, while the other variables could be measured. If prior to the test, gas were pulled from the UBA until the breathing bags were collapsed, Vci would be zero (to do this in U.S. Navy UBA, the diluent-add valve would have to be defeated). Vsumi would then be the sum of tubing volume (Vx) and initial syringe volume (Vsi), which would be known, and A could be calculated from the measured pressure (h) at the end of the tidal volume. (Vx should include the floodable volume of the gas filled hoses within the UBA). Knowing A, the test independent elastance (Eti) can be calculated as Eti = 1000/A where A is in cm^2, if only the water column effect is considered. In UBA, the floodable volume must be included in the elastance, so equation 17 could be used, where Vsumi = floodable volume of UBA with bag collapsed.

In Table 4, the last row of data was obtained with Vci = 0. Substitution

of that row's data into equation 20 yielded a value of 40 cm^2 for A, and an Eti of 25 cm/liter. These values of A and Eti were determined solely by the water column effect in the UBA. In the above examples, measured elastance (Etd) was as much as 21% lower than the test independent elastance (Eti).

4.2.2 Pressure Prediction with Divers

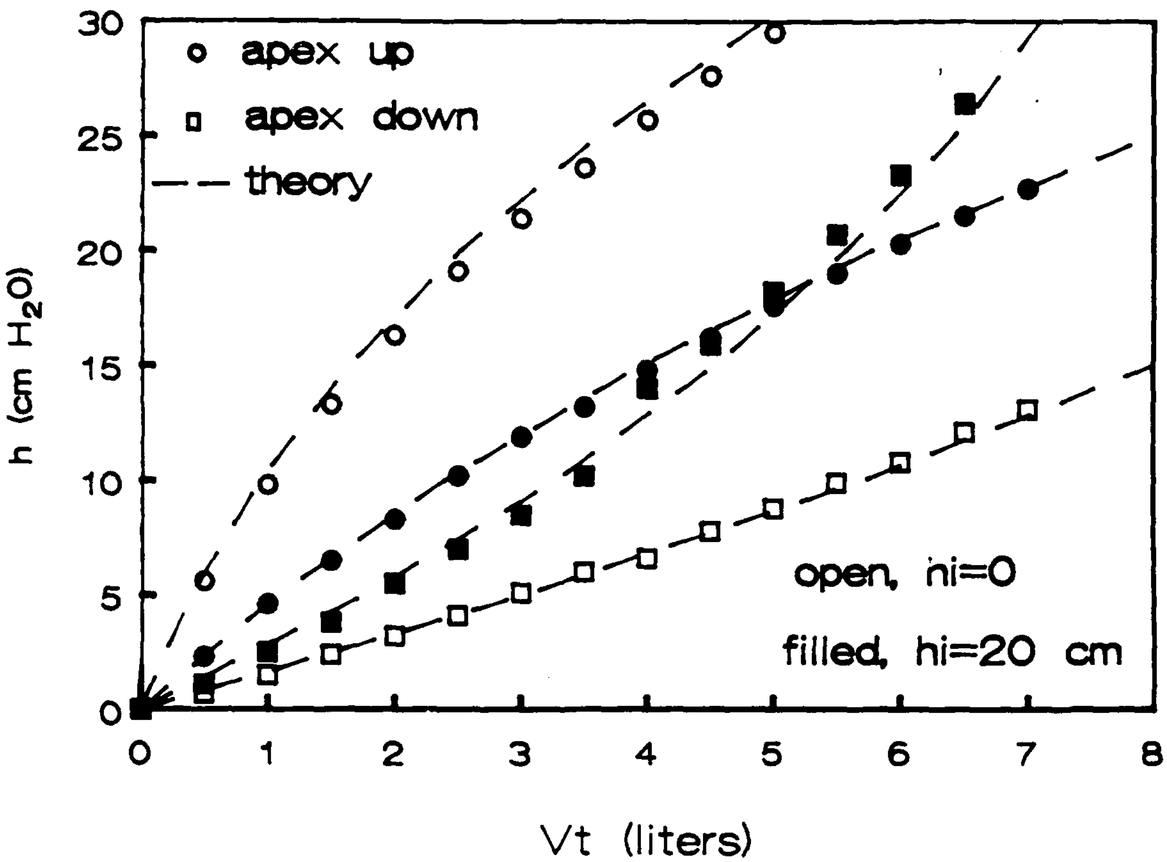
While the above correction is necessary for eliminating the effects of differing unmanned test setups, the test - independent elastance (Eti) is a worst case. The diver will never be subject to that elastance, because his respiratory tract contributes 4-5 liters to Vsumi. Furthermore, the gas-filled volume within the rig, excluding the breathing bag(s), also reduce the elastance to which the diver is exposed. Of course, knowing Eti, and making some assumptions about diver lung volume, various forms of equation 16 can be used to estimate the elastic pressures encountered by the diver.

4.3 Square-Base Pyramid.

Water column tests were also performed with a square-base pyramid whose top was cut off and capped to permit a hose connection. The pyramid dimensions are illustrated in Appendix B, Section 6. The results of these tests are shown in Fig. 3. The raw data are listed in Appendix F.

Fig. 3 illustrates the P-V characteristics of this truncated pyramid. Elastance was nonlinear for both apex up and apex down orientations. In the apex up position elastance decreased as tidal volume increased; in the apex down position elastance increased as tidal volume increased. The points generated by theory followed the experimental data almost exactly. The theoretical equations assumed Vh = Vt, and the initial volumes of other components were ignored.

FIG. 3. VERTICAL PYRAMID WATER COLUMN - COMPARISON OF THEORY AND EXPERIMENT



Similar nonlinearity with both convex and concave curvatures can be expected for the cone. The shape of the elastance curves can be altered by changing the design of these structures, for example, by changing the angle of the apex, the height, the size of the base, or by using curved rather than straight sides.

4.4 Elastic Balloon/Breathing Bag

4.4.1 Punch-Ball Balloon

Experiments on a punch-ball balloon were carried out in air in an 88.5 l plethysmograph according to the schematic in Fig. 4, in order to verify the mathematical model(s) in Appendix C. Two syringes were used, one to generate Vt and hence the balloon volume, and the other to absorb the balloon volume, thereby keeping the pressure inside the box equal to atmospheric. Two water manometers were used, one to measure internal pressure of the balloon, and the other to measure the internal pressure of the box.

The punch-ball balloon was used because it was almost spherical when inflated and thereby fit the model assumptions very well. Maximum size was about 14 liters. The minimum volume at which the balloon attained a spherical shape (V0) was between 0.36 and 0.42 liters obtained by caliper measurement and by extrapolation of the P-V curve, respectively. The initial volume of the syringe (Vsi) was 3 liters, and the tubing volume (Vx) was 0.5 liters. The data in tabular format are given in Appendix D. Figures 5 and 6 show the data graphically along with model predictions.

Figure 5 shows very good fit between the data and the three-zone, linearized model. The model parameters for zone II were m = 60 cm H_2O/liter and q = -18 cm H_2O; for zone III the corresponding parameters were m = -3.3 cm H_2O/liter and q = 37.9 cm H_2O.

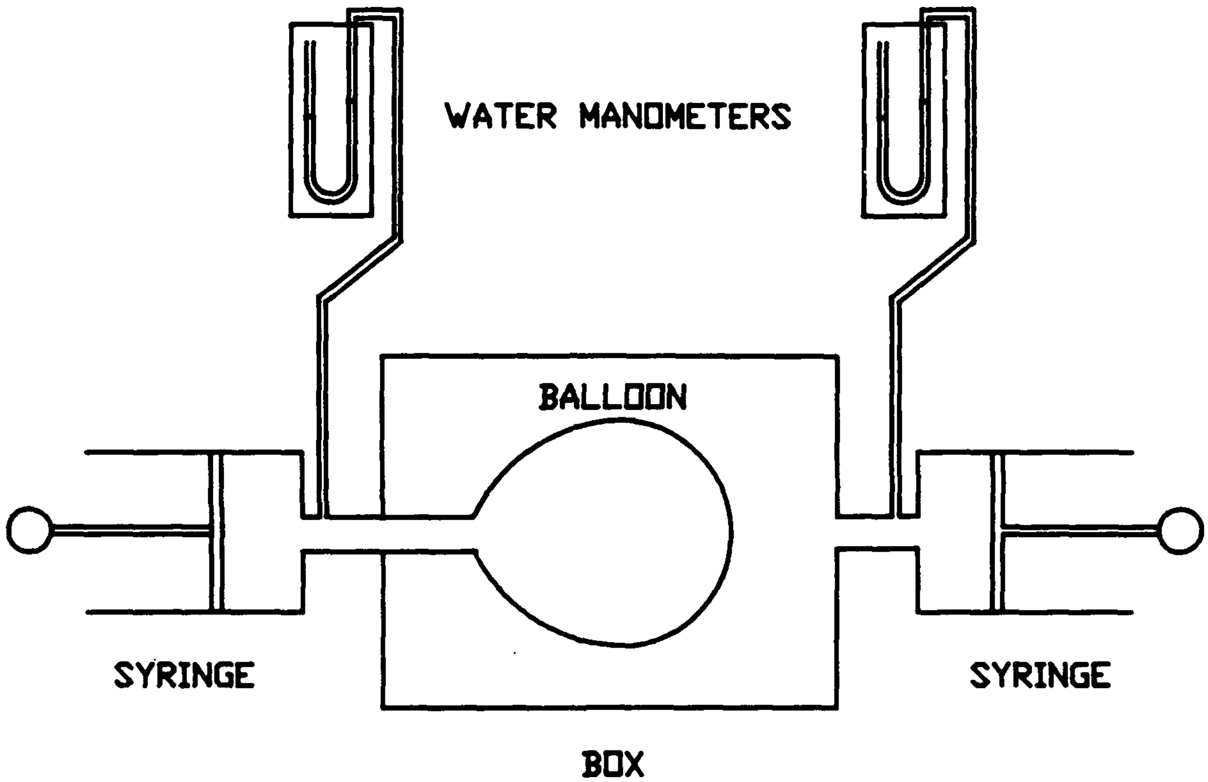


Fig. 4. Plethysmograph Apparatus for Measuring Volume of the Punch-Ball Balloon

FIG. 5. DATA AND 3-ZONE LINEAR MODEL FOR THE PUNCH - BALL BALLOON SYSTEM

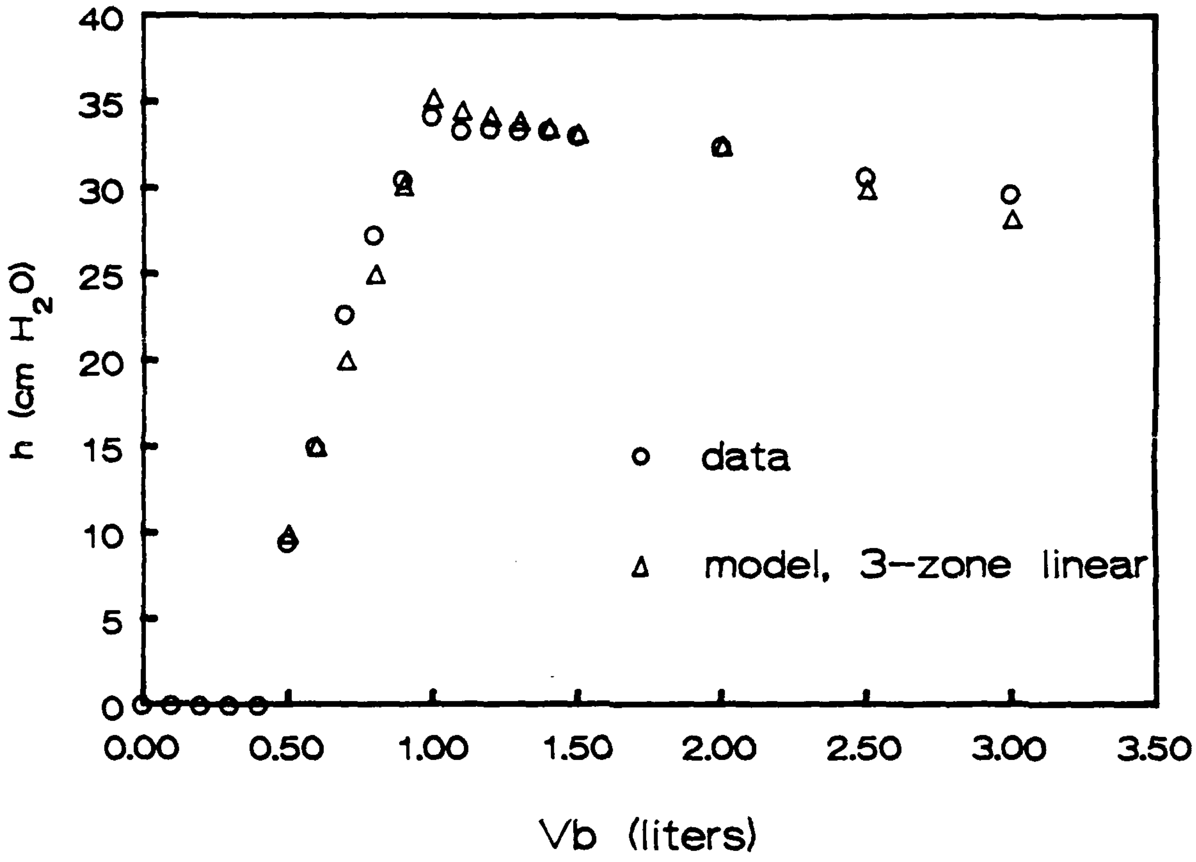


FIG. 6. THE ELASTIC BALLOON MODEL – CURVE FIT FOR C1 AND V0 PARAMETERS

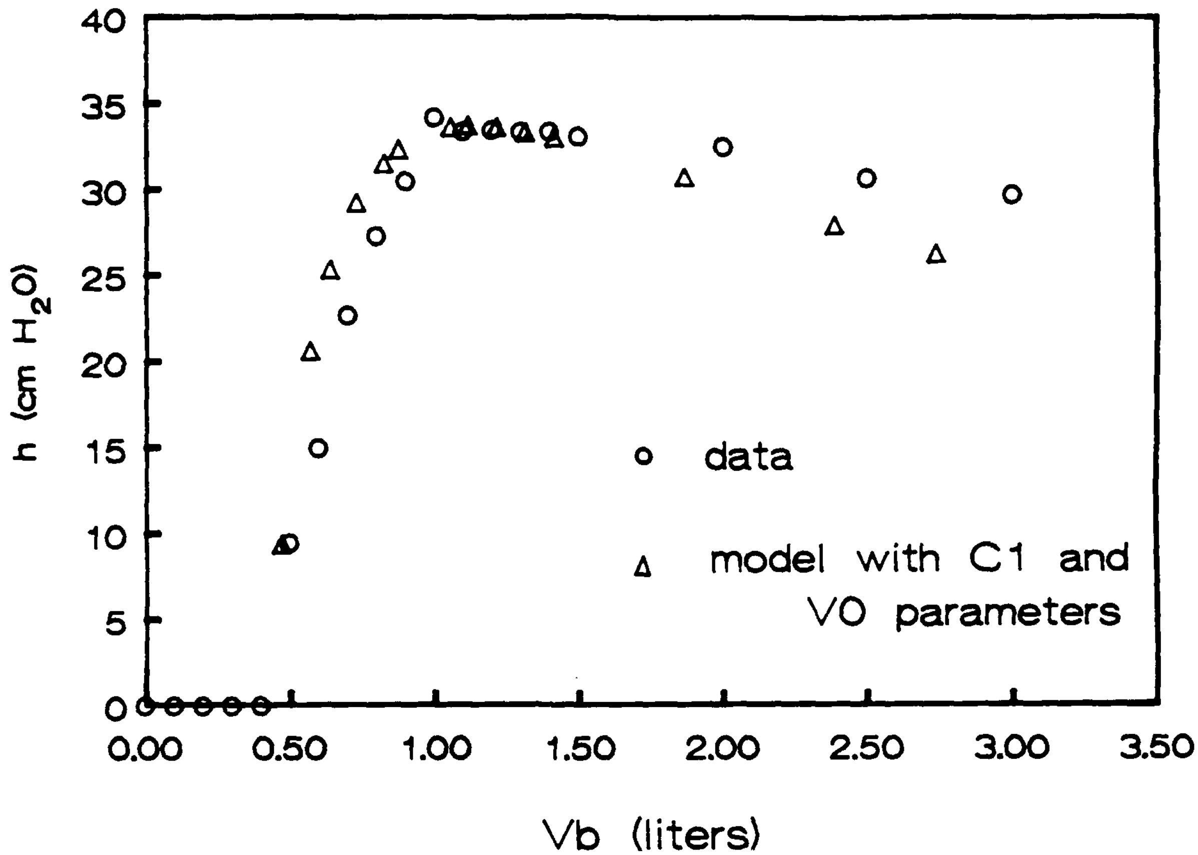


Figure 6 shows the balloon volume-pressure curve to test the balloon elasticity model (the balloon volume is not the same as Vt in Fig. 4). Since the original volume of rubber and the elastic modulus of the rubber were not known, a two-parameter curve fit was employed to get the model constants. The best fit was obtained using points on each side of the maximum [(.467, 9.46) on the positive slope region and (1.41, 33.1) on the negative slope region]. The agreement with the other experimental points was excellent, except at large volumes where the elastic modulus may not have remained constant as assumed. The predicted maximum pressure and negative elastance are both clearly demonstrated.

4.4.2 Weather Balloon

A weather balloon is an extreme example of an enclosed system with low elasticity. Figure 7 shows the sigmoidal P-V curve for a weather balloon. Neither the pressure maximum nor a negative elastance was obtained with this balloon, in contrast to the punch-ball. The elastance was near zero except in the zone II region between V0 and Vpmax. Here the elastance was 0.238 cm H_2O/liter, considerably lower than the punch-ball balloon. This balloon was much larger than the punch ball and had a much more flaccid and stretchable feel to it.

4.4.3 Breathing Bags

Data from two anesthesia breathing bags of different sizes are shown in Figure 8. Both show "limp balloon" characteristics up to their respective V0's, then strong resistance to high pressures (over 16 cm H_2O). The elastance of the 5-liter bag in zone II was 16 cm H_2O/liter; that of the 15-liter bag in zone II was 6 cm H_2O/liter. In the case of the 15-liter bag,

FIG. 7. PRESSURE - VOLUME DATA FOR RED WEATHER BALLOON

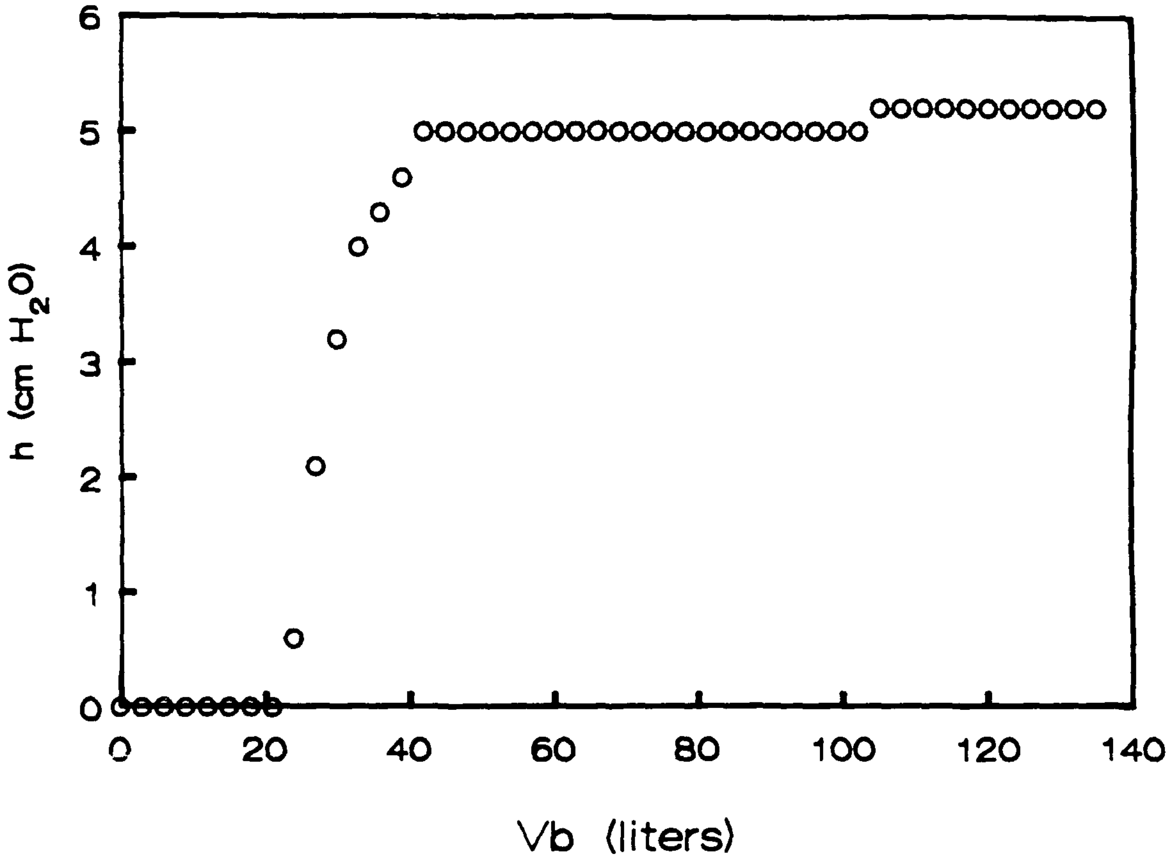
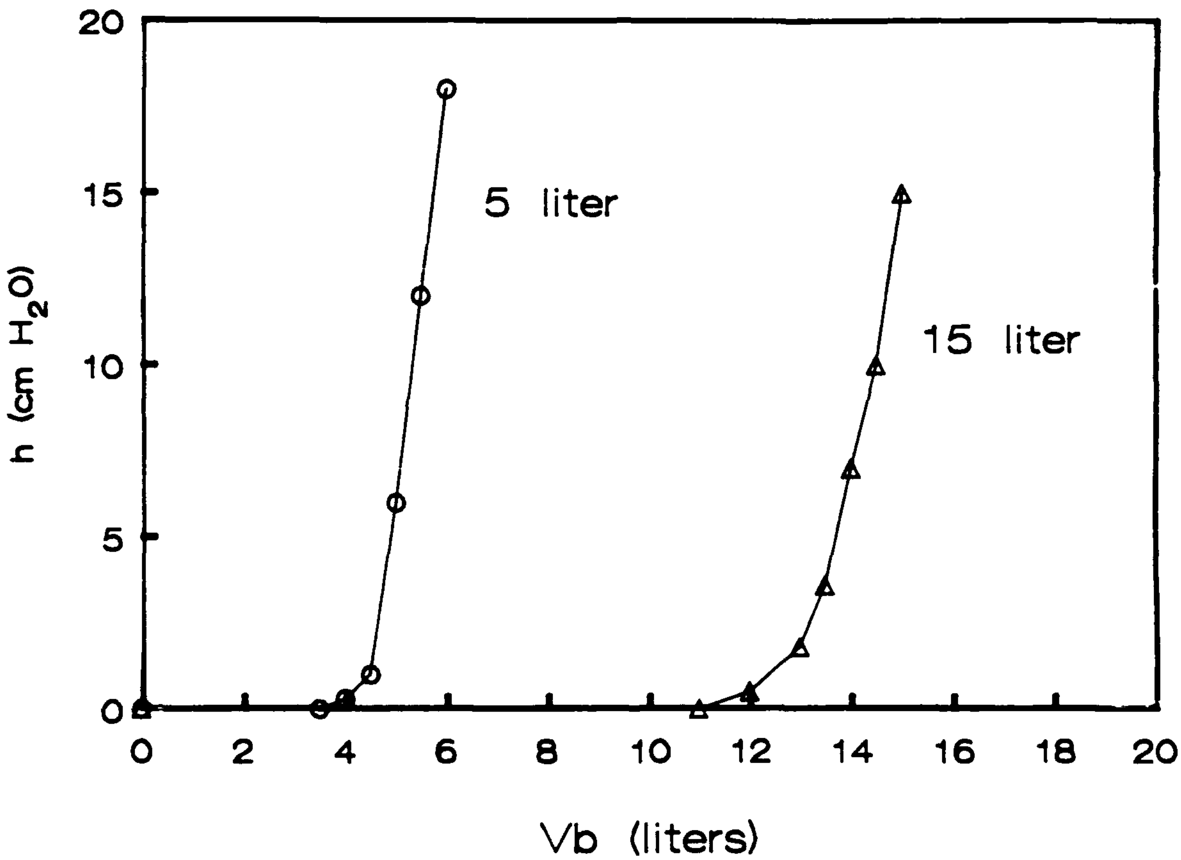


FIG. 8. PRESSURE - VOLUME DATA FOR
5 - and 15 - LITER BREATHING BAGS



zone II occurred just outside its rated capacity. Both bags showed zero elastance in zone I.

4.5 Elastic Loading in the MK-15

The elastance of a MK-15 closed-circuit UBA was evaluated in unmanned tests in both air and water. The breathing bag in the MK-15 is a large diameter, cylindrically shaped, rubber elastic container. The far end of the cylinder is closed by an inflexible diaphragm. The bag expands by unfolding pleats in the rubber around the sides of the cylinder, somewhat like an accordion or bellows. These folds would occasionally stick, producing transient unevenness in the bag expansion and corresponding scatter in the P-V data. When the bag filled the housing and could not expand further, a continued increase in pressure resulted in the unseating of a pop-off valve, dumping excess gas and relieving internal pressure.

In the air, the elastic characteristics of this breathing bag would be expected to follow the elastic bag model. In the water, however, the bag expansion generated elastic loads due to the moving air-water interface, and was best modeled by the cylindrical water column.

The orientation of the MK-15 underwater affects its elastance. In the following experiments, four orientations of the rig were investigated; two horizontal positions (one oriented as if worn by a diver facing up and a second oriented as if the diver were facing down), a vertical position, and a 45° angle facing down.

In the air the bag assumed an equilibrium configuration depending on its weight and the orientation of the rig. The bag rested on the canister when the rig was horizontal - face up. In this position the bag was essentially collapsed. When the rig was horizontal - face down, the bag hung almost to

the housing. On its side, the bag assumed a mid-way position. In the water all runs started with the bag fully collapsed, since hydrostatic pressure forced the diaphragm up against the canister.

The results of these experiments are shown in Figures 9 and 10, with Fig. 10 being a repeat run of the experiment on a different day. Both graphs show the same basic trends and almost identical data. In the water with the rig in the horizontal position the elastance was essentially the same for face up and face down positions, which would be expected if elastance was governed by a water column. The elastance in the two horizontal positions were 1.8 cm H_2O/liter, which was calculated based on a "best fit" slope of the data in the middle region of tidal volume. In the extreme collapsed or extreme inflated conditions, bag elastance was affected by factors other than the water column. The absolute values of the pressures in Figs. 9 and 10 reflect the difference in the depth to which the rig was submerged. The pressures for the face up position were lower than those of the face down position, because the breathing bag was shallower in the water when the rig was face up. The position of the rig support was not changed to adjust the depth of submersion in these instances. Depth of submersion, therefore, had no effect on the elastance.

The data in Figs. 9 and 10 reflect tests which started with the bag fully collapsed and ended with the bag fully inflated. In reality the diver breathes around a mid-point of bag distention, approximately 2.5 liters in Figs. 9 and 10. Thus, the elastance of the rig in practice would be linear, except at the extreme collapsed and extreme inflated positions, where the data show pressure increasing rapidly for small volume increments. This is largely due to the bag having inflated or collapsed to its limit. The curves are not symmetric at the ends, because the pop-off valve action at full distension has

FIG. 9. ELASTANCE DATA FOR MARK 15 – RUN 1

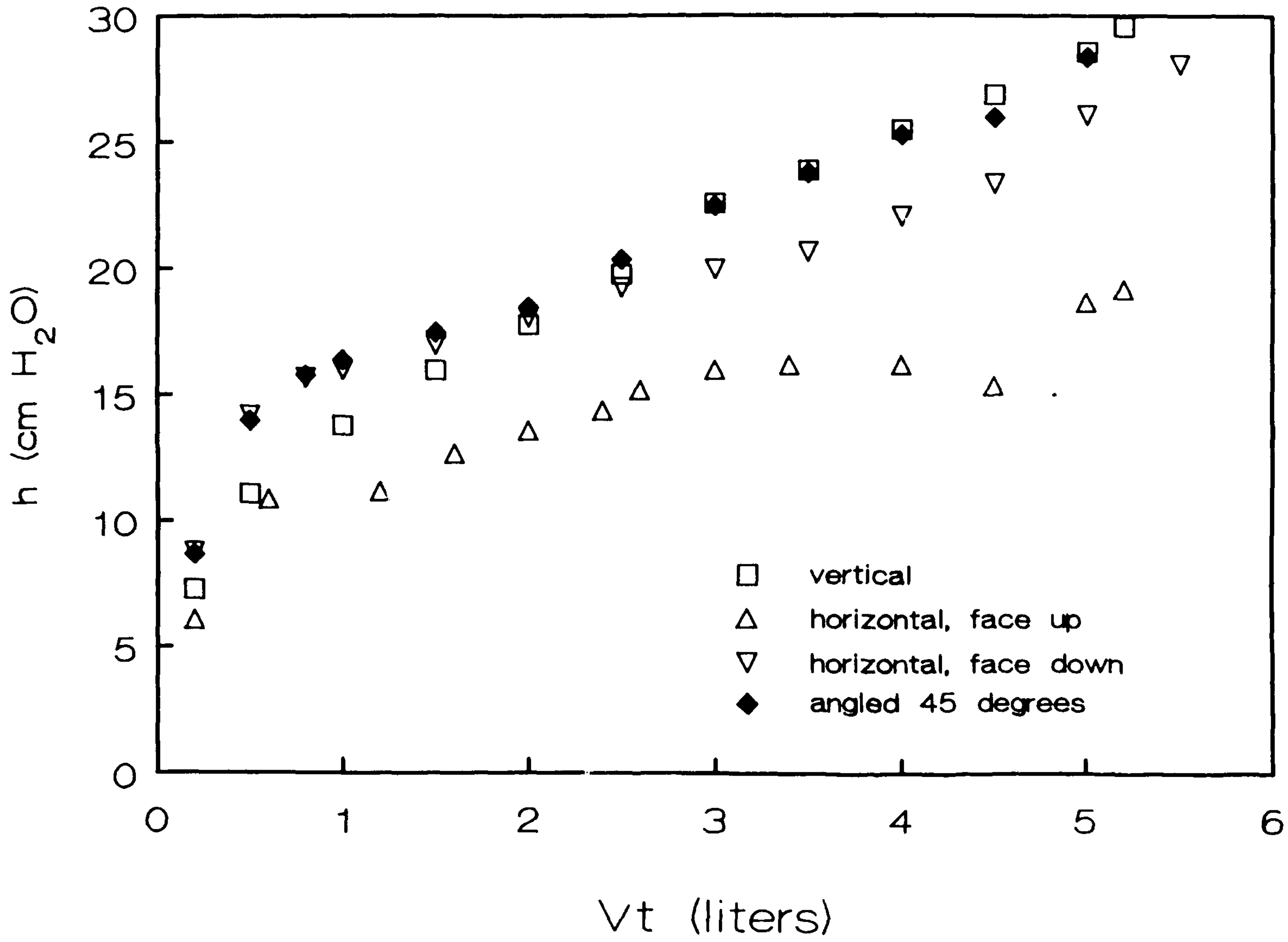
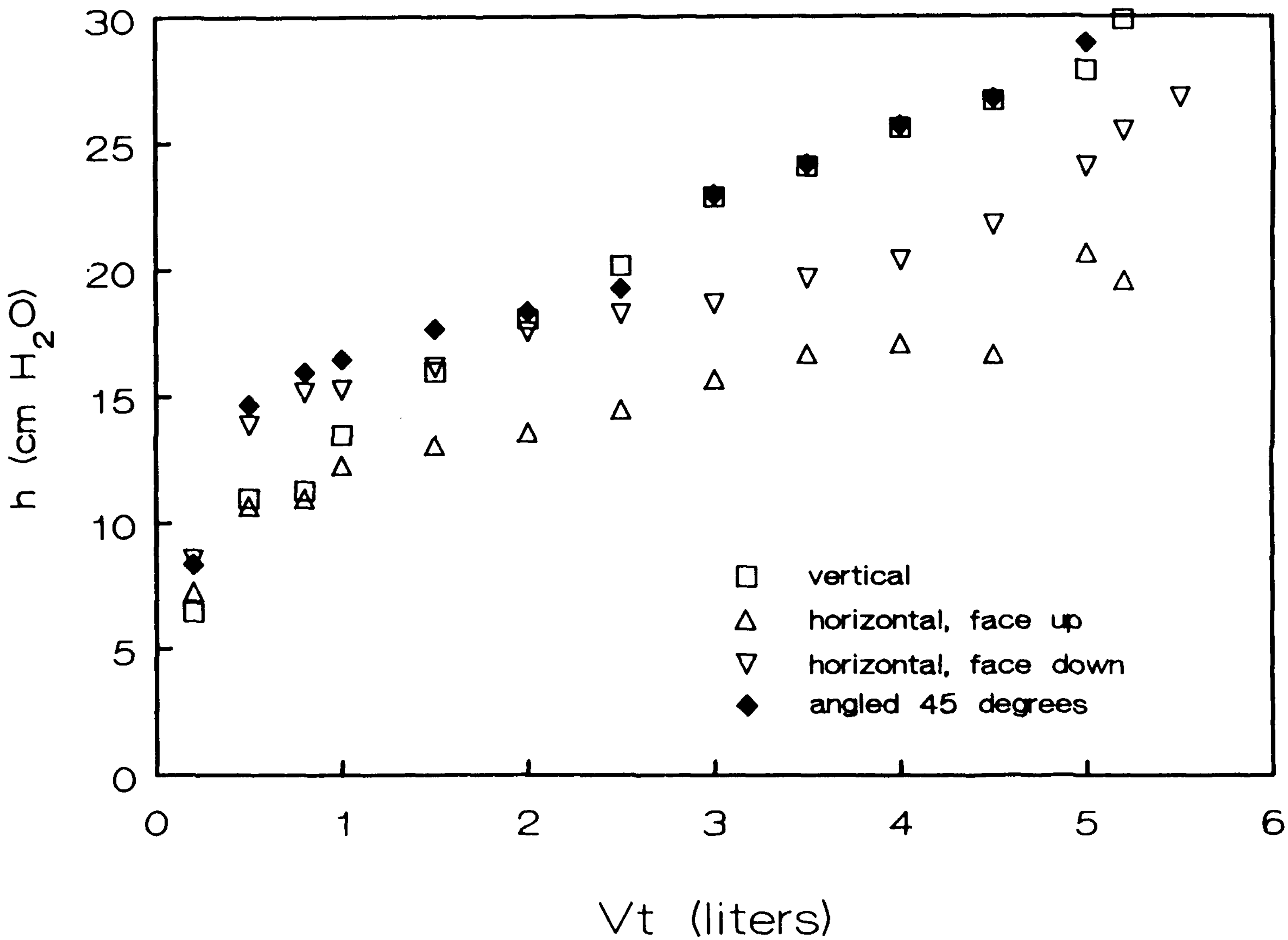


FIG. 10. ELASTANCE DATA FOR MARK 15 – RUN 2



no counterpart when the bag is fully collapsed.

In the vertical position, elastance was about 4.0 cm H_2O/liter; in the horizontal position, it was about 1.8 cm H_2O/liter. In the angled position the elastance was more complex, reflecting the horizontal elastance at small tidal volumes and the vertical elastance at higher tidal volumes. The switch-over point seemed to be 2.5 liters in both Figs. 9 and 10. In practice this means a diver in this position could experience different elastic loads during a breath, depending upon the initial volume of the bag.

The elastance characteristics of the angled position in the water may be explained by an uneven expansion of the breathing bag according to the sequence of events shown in Fig. 11. In the angled position, the right (top) corner of the bag in Fig. 11 expands first until the rigid diaphragm is horizontal (step 1). The diaphragm probably slides to the right somewhat as well as rotates. The left corner slides downward along the top of the canister. Apparently this does not take much volume; the data in this period cannot be differentiated much from the horizontal data. Once the diaphragm is horizontal, a classic water column effect (step 2) is experienced until the right corner of the bag contacts the housing. This can occur with the right corner continuing to hug the side wall or moving vertically as shown. When the right corner can no longer move vertically, the left corner of the bag moves up (step 3) until the housing is fully contacted (this motion is probably another sliding/rotating motion of the diaphragm). At this point the bag is fully inflated and further increases in pressure are produced by gas compressibility (rigid container effect) until the pop-off valve releases.

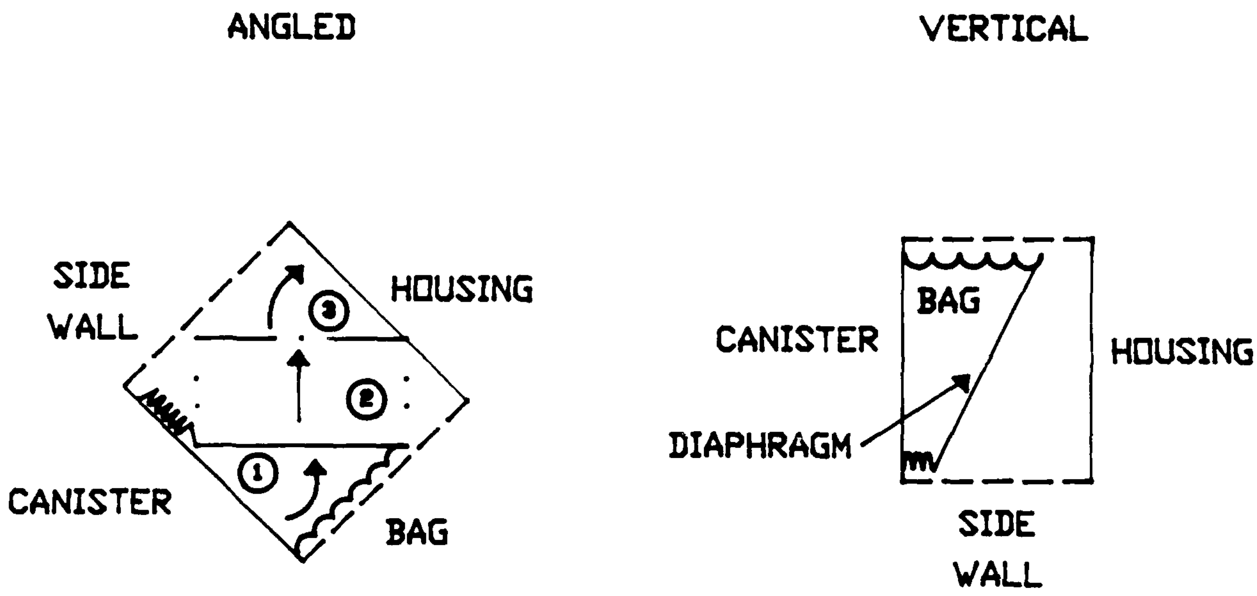


Fig. 11. Expansion of Mark 15 Bag in Angled and Vertical Positions in the Water

In the air, a limit of 4 - 5 cm H_2O was reached in all cases. This was the "cracking pressure" or the pressure at which the pop-off valve released. Generally, limp balloon behavior was shown until the bag expanded to the housing limit, then pressure increased rapidly, probably due to the box elastance rather than rubber stretching, although both may play a part. The change in behavior occurs at different Vt's, because the bag position (hence initial volume) varied according to orientation. Tabulated data can be found in Appendix E.

The results for the MK-15 could be predicted from a water column analysis, once the dimensions of the rig were known. Measurements were taken on the rig, and the water column model in its simplified form was used to calculate a predicted elastance. In a horizontal position the cross-sectional area of the bag was about 731 cm^2, resulting in an estimated elastance of 1.37 cm H_2O/liter. In the vertical position the cross section is no longer circular, but rectangular. The estimated dimensions of 7.6 x 30.5 cm (3 inches x 12 inches) gives an area of 232 cm^2. This results in an elastance of 4.3 cm H_2O/liter when only the area term is used (equation 10) and 4.1 cm H_2O/liter when the other volumes are included (equation 9). These compare very well with the slopes of the appropriate lines (1.8 and 4.0 cm H_2O/liter are the slopes for horizontal and vertical orientations, respectively) in Figures 9 and 10. In the horizontal position, if fewer points from the extremes are used, the slope becomes approximately 1.4, which improves the agreement.

5.0 SUMMARY

1. Mathematical models based upon Boyle's law were defined for various types of elastic loads. Measurements of elastance in tests of physical models agreed well with theoretical predictions, thus demonstrating the validity of this approach. Knowing the elastance of various gas-filled geometrical shapes the elastance of any arbitrary enclosed space underwater can be predicted.

2. In the elastic loading literature, fixed elastances are typically generated by rigid boxes (5,6). The method commonly used for calculating the elastance of those boxes (eqn. 11) lacks general validity. It is only an approximation. A more precise, general formula (eqn. 9) is presented. Both simplified and general equations are given for other structures as well, such as the water column and the elastic breathing bag.

3. The MK-15 UBA has elastic load characteristics that can be accurately described by a water column model when the rig is immersed. When dry, the MK-15's elastance is similar to that of the elastic bag and the box, as might be expected from its structure. Since elastance may be altered by UBA orientation, it should be measured under the conditions in which the UBA is to be used.

4. The measurement of elastance is dependent upon the volume characteristics of various unmanned testing equipment. Therefore, the elastance of a UBA determined by different laboratories may be test dependent. A method was presented for determining a test-independent elastance (Eti) for a case in which the elastance was primarily due to a water column effect. This method should also be applicable to measuring other forms of elastances. The volume characteristics of the test apparatus must be known in all cases. Elastic pressures likely to be encountered by a diver would be less than Eti measured in unmanned testing, and can be found by including lung volume into the system volume of the governing equation.

5. The effect of nonlinear elastances on underwater exercise performance needs to be addressed. If high peak pressures near the end of expiration prove undesirable, then an expiratory elastance that decreases as a diver exhales (as in the pyramid or cone, apex up) might prove beneficial. On the other hand, relatively high airway pressures at the end of expiration may serve some physiological benefit, especially during immersion. That possibility could be tested by the use of pyramidal gas spaces oriented apex down.

This work should best be viewed as a primer on UBA elastance. Although we have attempted to be comprehensive in the mathematical treatment of elastance, many practical aspects have yet to be addressed. Undoubtedly, suggested methods for measuring and working with elastance will need modification as workers gain experience. This work delineates the issues, aids in unmanned testing, and suggests tools for physiological studies of elastance.

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APPENDIX A1: SUMMARY OF SIMPLIFIED FORMULAS FOR ELASTIC LOADING

Box

$$h = \frac{1033}{Vbox + Vx + Vsi} Vt \cong [1033/Vbox]\ Vt$$

Note: 1033 = Pi = atmospheric pressure, cm H_2O

Water Column

1. Vertical Cylinder

$$h = \frac{1}{A/1000 + (Vci + Vx + Vsi)/1033} Vt$$

$$h \cong [1000/A]\ Vt$$

$$h \cong [4000/\pi\ D^2]\ Vt$$

where D is in cm, A is cm^2, V is liters and h is cm H_2O.

2. Cone, Apex Up

$$h \cong [12000\ H^2/\pi\ D^2]^{1/3}\ Vt^{1/3}$$

3. Cone, Apex Down

$$h \cong [4000/\pi\ D^2]\ Vt$$

These formulas for the cone are likely to be oversimplified. The full equations for the cone are given in Appendix B, and the full equations for the pyramid are given below.

4. Square-Base Pyramid, Apex Up

$$Vt = (B^2/3H^2)(h/1000)[h^2 + 3h(hi + zc) + 3(hi + zc)^2]$$

5. Square-Base Pyramid, Apex Down

$$Vt = (B^2/3H^2)(h/1000)[h^2 - 3h(H - hi) + 3(H - hi)^2]$$

where B, H, hi, zc are all in cm and Vt is in liters.

APPENDIX A2: VERTICAL CYLINDER WATER COLUMN ELASTANCE - SUMMARY

In order to more easily select or design for elastic loading, the simplified equations may be plotted in various forms to show the dependency on factors of interest. Figures 12 - 14 show this information. Figures 12 and 13 show the effect of cylinder diameter on pressure-volume relationships and on elastance. In these computations both the A term and the Vsumi terms were used, and Vsumi was taken to be 7 liters. The plots are self-explanatory. If a certain elastance is desired, the diameter of the cylinder that would supply that can be found by reading across to the line (in Fig. 12, for example) and down to the diameter. The reverse is just as easy.

Figure 14 shows the difference between elastance predicted by the two simplified formulas, one with only the A term (equation 18) and the other with both the A and Vsumi terms (equation 17). Again Vsumi was taken to be 7 liters, and the Vt term in this expression was neglected. This plot shows where the most simplified form may be used with little error. At an elastance of 8 cm H_2O/liter, the deviation is 2.5%, and below 8 cm H_2O/liter the deviation goes to zero. At an elastance of 12 cm H_2O/liter, the deviation is 8.3%, and at 15 cm H_2O/liter it is over 10%.

FIG. 12. VERTICAL CYLINDER MODEL:
SIZE-PRESSURE-VOLUME CURVES

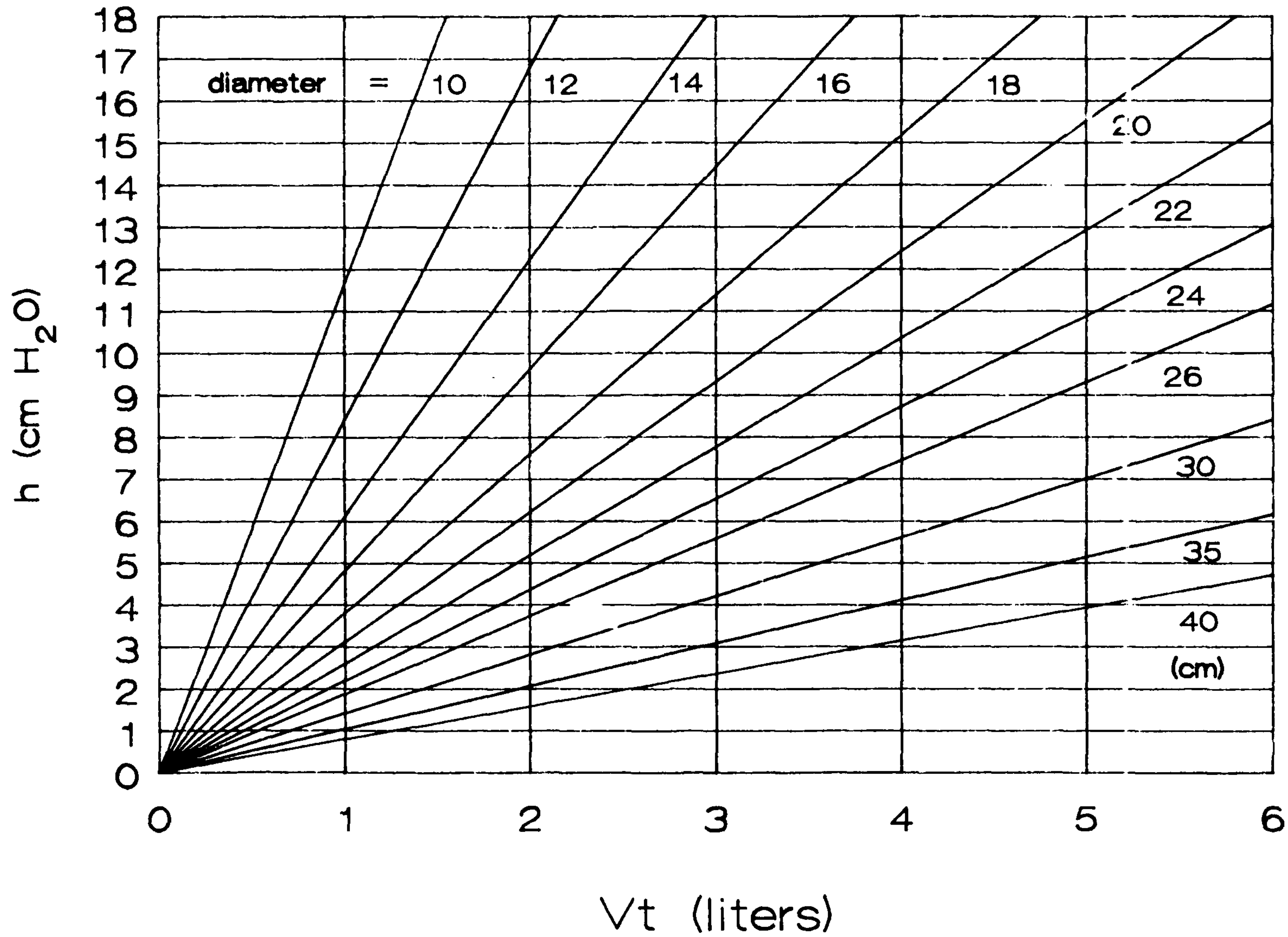


FIG. 13. VERTICAL CYLINDER MODEL PREDICTIONS: ELASTANCE vs. SIZE

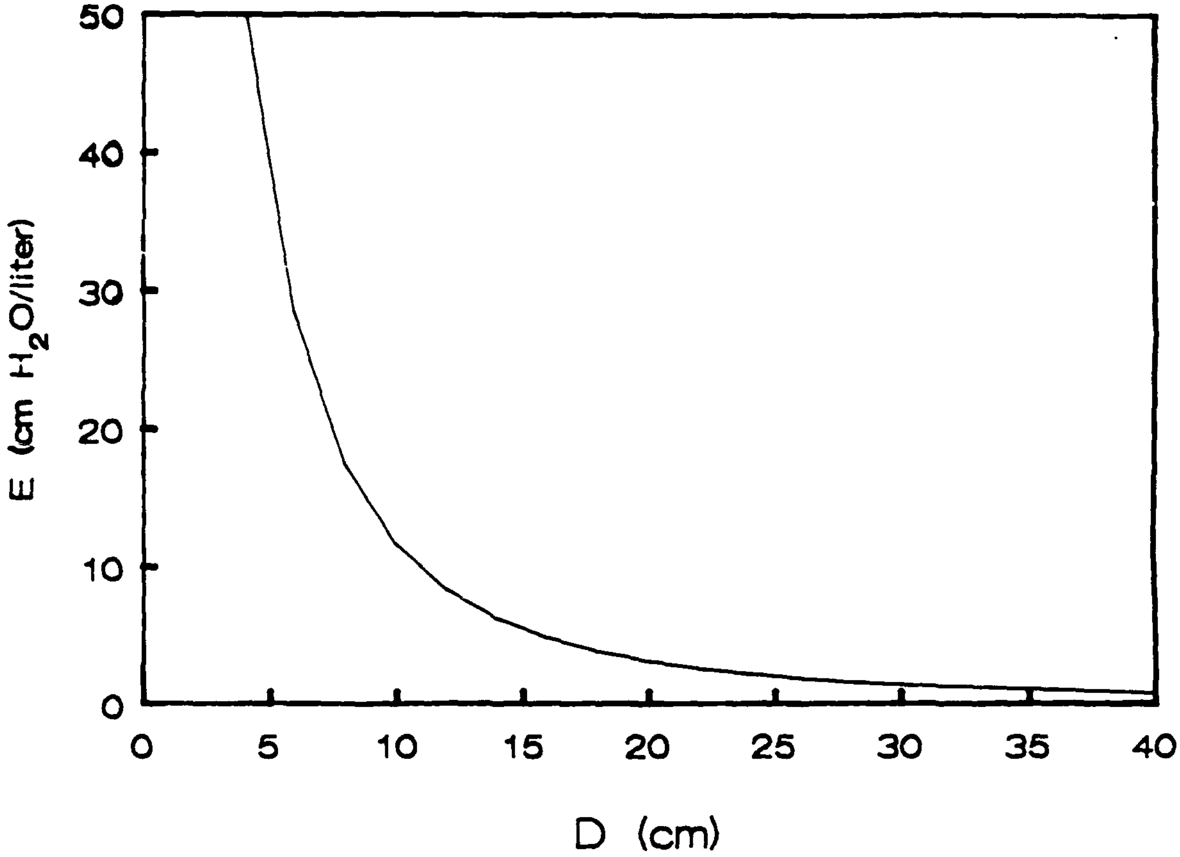
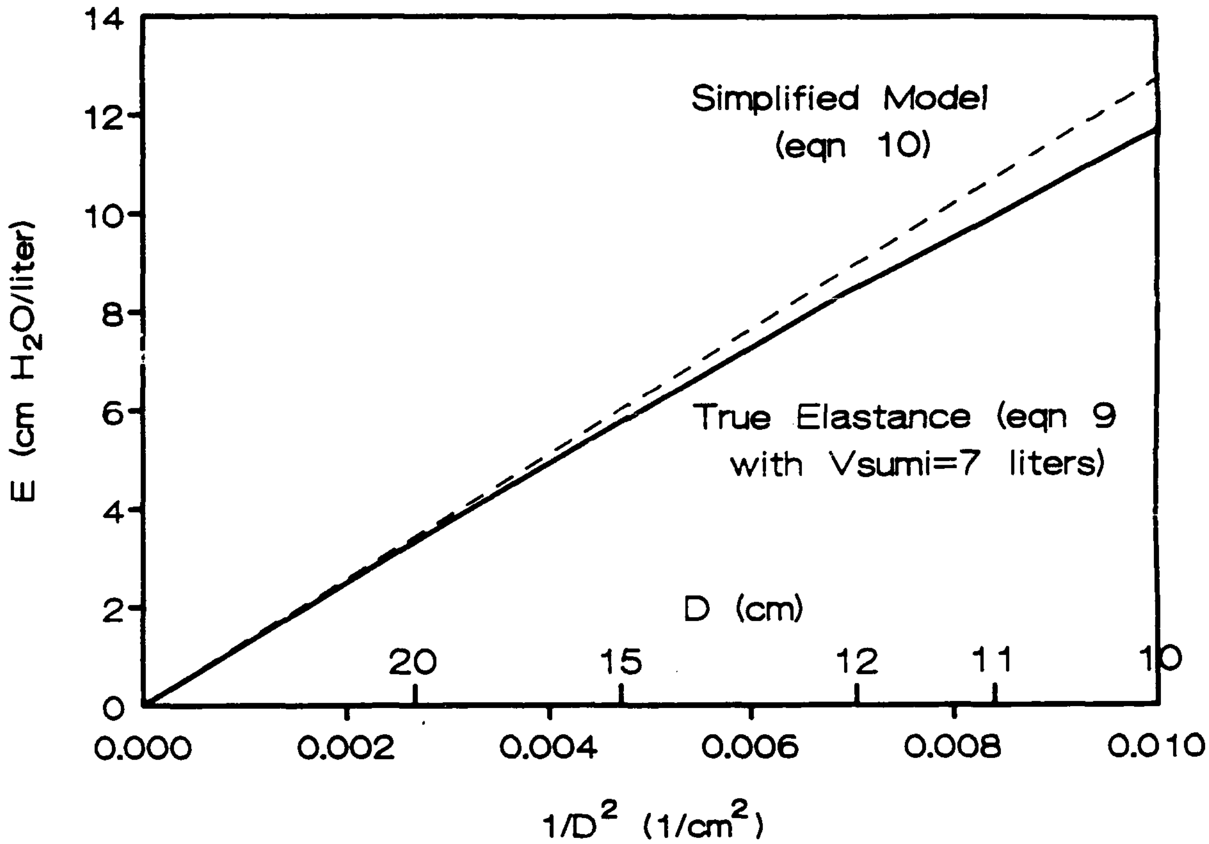


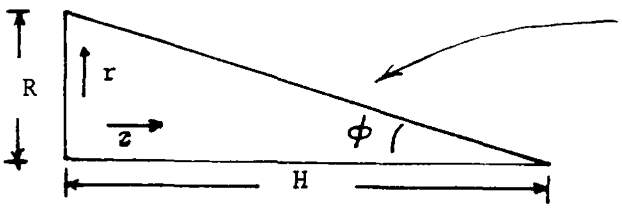
FIG. 14. SIMPLIFIED MODEL AND TRUE ELASTANCE FOR THE VERTICAL CYLINDER



APPENDIX B: CONE AND PYRAMID FORMULA DERIVATIONS

1. General Formula for Volume of a Cone

The general formula for the volume of a cone can be derived from various considerations. For example, the diagram shown below can be used to develop the volume formula.

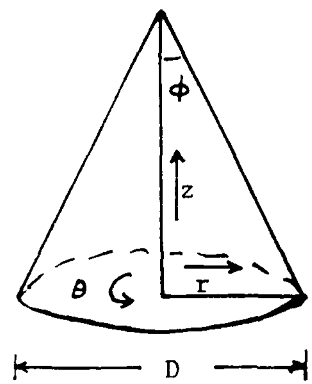


$$z(r) = (R - r)/\tan\phi$$

at $r = 0$, $z(r) = H$

at $r = R$, $z(r) = 0$

$$\begin{aligned} V &= \int_0^R \int_0^{2\pi} z(r)\,r\,d\theta\,dr \\ &= 2\pi \int_0^R [(R-r)/\tan\phi]\,r\,dr \\ &= (2\pi/\tan\phi)(R^3/2 - R^3/3) \\ &= \pi R^3/3\tan\phi \end{aligned}$$



which can be rearranged into the formula

$$V = \pi H^3 (\tan\phi)^2/3$$

or

$$V = \pi D^2 H/12$$

where H is the axial length of the cone, base to apex, and D is the base diameter. The cone volume is one-third the volume of a cylinder of diameter equal to the base of the cone. The formulas above are related by

$$\tan\phi = R/H = D/2H$$

where $R = D/2$, and ϕ is the cone half angle at the apex.

2. Frustum of a Cone

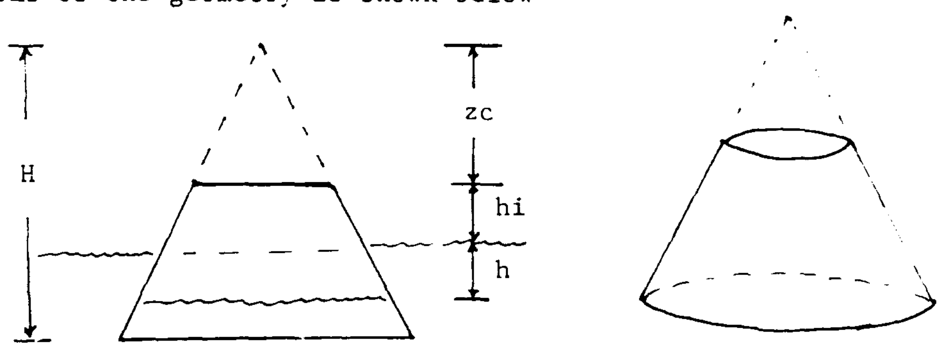
The actual geometry of the volume likely to be used for testing is that of a frustum of a cone with two flat faces perpendicular to the main axis at each end of the conical surface. This is to allow for tubing and other connections to be made. The formula here, too, can be derived directly or looked up. It is

$$V = \pi\ (D^2/12H^2)(z)(3H^2 - 3Hz + z^2)$$

where z is the height of the frustum, base to parallel face. Either method can be used to derive the general pressure - volume characteristics for the cone shape as we have it. An alternative derivation is given later.

3. Cone Apex Up Derivation

For the actual geometry, the volume equation must be reformulated in terms of the geometry as shown below



The cap volume down to zc is non-existent; the volume due to the hi dimension is Vci, and the volume added by water column movement (Vh) corresponds to the h dimension. The volume of the frustum of the cone is given by subtracting the cap volume from the total volume of the cone (to apex). If the water in the column goes above the rest position, the sign on Vh would be negative.

Assuming $Vt \cong Vh$, which follows from $h << 1033$, we have,

$$Vt = Vcone - Vci - Vcap = Vcone - (Vci + Vcap)$$
$$= [\pi D^2/12H^2][(h + hi + zc)^3 - (hi + zc)^3]$$

This requires some algebra to get a workable formula. First,

$$(hi + zc)^3 = hi^3 + 3hi^2(zc) + 3hi(zc^2) + zc^3$$

This gets subtracted from the first term, which can be multiplied out in steps,

$$(h + hi + zc)^2 = h^2 + 2h(hi) + 2zc(h) + 2zc(hi) + hi^2 + zc^2$$

Multiplying this by $(h + hi + zc)$ gives the following 18 terms:

$$\begin{aligned}(h + hi + zc)^3 = h^3 &+ 2h^2hi + 2h^2zc + 2zchih + h(hi^2) + hzc^2 \\ &+ h^2hi + 2hhi^2 + 2zchih + 2zchi^2 + hi^3 + zc^2hi \\ &+ h^2zc + 2hihzc + 2zc^2h + 2zc^2hi + hi^2zc + zc^3\end{aligned}$$

which can be consolidated to,

$$\begin{aligned}&= h^3 + hi^3 + zc^3 + 3zchi^2 + 3hi^2h + 3h^2hi \\ &+ 3zch^2 + 3zc^2h + 3zc^2hi + 6zchih\end{aligned}$$

Subtracting the first cubic expression gives

$$Vt = [\pi D^2/12H^2][h^3 + 3h^2(hi + zc) + 3h(hi + zc)^2]$$

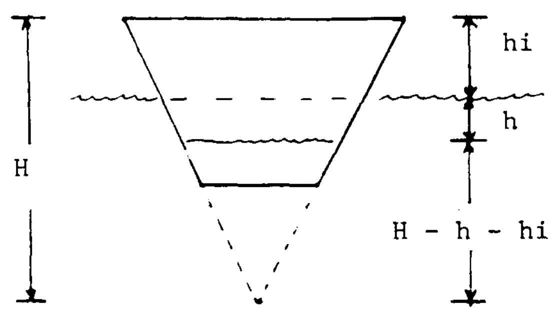
or,

$$Vt = [\pi D^2/12H^2](h)[h^2 + 3h(hi + zc) + 3(hi + zc)^2]$$

This expression is not easy to simplify further, since the parameters, h, hi and zc are all the same order of magnitude. However, if hi = 0, and h is greater than zc, then as an approximation, Vt is proportional to h^3, or h is proportional to the cube root of Vt, and the elastance is very nonlinear.

4. Cone Apex Down Derivation

In this case the cap is unimportant, and hi is measured from the base as shown below.



The volume of interest is Vh + Vci, which is given by subtraction of the appropriate cone volumes, the total cone minus the cap, where the cap height is by difference (H - h - hi) rather than zc. Thus,

$$Vnet = VH + V(H - h - hi)$$

But,

$$Vt = Vnet - Vci = (VH - V(H - h - hi)) - (VH - V(H - hi))$$

$$Vt = V(H - hi) - V(H - h - hi)$$

$$Vt = [\pi D^2/12H^2][(H - hi)^3 - (H - h - hi)^3]$$

and $$(H - hi)^3 = H^3 - 3H^2hi + 3Hhi^2 - hi^3$$

The other cubic is done in steps.

$$(H - h - hi)^2 = H^2 + h^2 + hi^2 - 2Hh - 2Hhi - 2hih$$

And then

$$\begin{aligned}(H - h - hi)^3 = H^3 &+ Hh^2 + Hhi^2 - 2hH^2 - 2hiH^2 + 2hhiH \\ &+ 2Hh^2 - hH^2 + 2hhiH - h^3 - hi^2h - 2h^2hi \\ &- hi^3 + 2hi^2H - hiH^2 + 2hhiH - 2hhi^2 - h^2hi\end{aligned}$$

Which reduces to

$$\begin{aligned}(H - h - hi)^3 = H^3 &- h^3 - hi^3 + 3h^2H + 3hi^2H - 3H^2h - 3H^2hi \\ &+ 6hhiH - 3hi^2h - 3hih^2\end{aligned}$$

Substituting, eliminating and rearranging gives

$$Vt = [\pi D^2/12H^2][h^3 - 3h^2(H - hi) + 3h(H - hi)^2]$$

Further simplification can be done, because H is always greater than either h or hi. If the H^2 term in brackets dominates,

$$Vt = \pi D^2 h/12$$

Thus, as an approximation, the elastance would be linear, basically the inverse of the cone base area. This would be the same as a cylinder with the cone base diameter. However, this is likely to be oversimplified (except for small h), because H is about the same order of magnitude as h.

5. Alternative Derivation for Cone - Apex Down

Instead of cone subtraction as the method of approach, the volume for a frustum of a cone can be written directly in integral form starting with $z = 0$ at the base. The radial distance is a function of z, therefore,

$$V = \int_0^z \int_0^{2\pi} \int_0^{r(z)} r\,dr\,d\theta\,dz$$
$$= \pi \int_0^z r^2 dz$$

where

$$r = R - z \tan\phi \quad \text{and} \quad \tan\phi = R/H \text{ as before.}$$

Therefore,

$$V = \pi \int_0^z (R - z \tan\phi)^2 dz$$
$$= \pi \int_0^z [R^2 - 2zR \tan\phi + z^2(\tan\phi)^2]dz$$
$$= \pi\ [R^2 z - Rz^2 \tan\phi + z^3(\tan\phi)^2/3]$$
$$= \pi\ z[3R^2 - 3Rz \tan\phi + z^2(\tan\phi)^2]/3$$

But $R = H \tan\phi$, and

$$V = (\pi\ z/3)[3H^2(\tan\phi)^2 - 3Hz(\tan\phi)^2 + z^2(\tan\phi)^2]$$

Thus,

$$V = (\pi D^2/12H^2)(z)(3H^2 - 3Hz + z^2)$$

The water column volume, assuming hi = 0 is,

$$Vh = Vt = V(h + hi) - Vhi$$

or,

$$Vt = (\pi D^2/12H^2)[(h + hi)(3H^2 - 3H(h + hi) + \\ + (h + hi)^2 - hi(3H^2 - 3Hhi + hi^2)]$$

Multiplying terms and cancelling,

$$\begin{aligned} Vt &= (\pi D^2/12H^2)[3H^2h - 3Hh^2 - 3Hhih + h^3 + 2hih^2 \\ &\quad + hhi^2 + 3H^2hi - 3Hhhi - 3Hhi^2 + hih^2 \\ &\quad + 2hi^2h + hi^3 - 3H^2hi + 3Hhi^2 - hi^3] \\ &= (\pi D^2/12H^2)[h^3 - 3h^2(h - hi) + h(3H^2 - 6Hhi + 3hi^2)] \\ &= (\pi D^2/12H^2)[h^3 - 3h^2(H - hi) + 3h(H - hi)^2] \\ Vt &= (\pi D^2/12H^2)(h)[h^2 - 3h(H - hi) + 3(H - hi)^2] \end{aligned}$$

which is identical to that derived before. Further simplification yields the same answer as before, also.

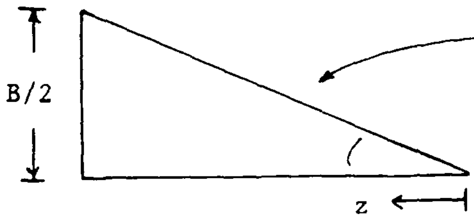
6. Square - Base Pyramid (Apex Up)

The frustum of a cone was too difficult to build in the time period available, but a similar shape - the square- based, four-sided pyramid was easier. Therefore, it would be of interest to know the corresponding formulas for this shape in the apex up and apex down positions. The development of the equations follows that of the frustum of a cone.

The volume can be computed from the integral of the base area (a function of height) times the differential height.

$$V = \int_0^H A(z)dz$$

where, A = length x width. From the diagram below, (here z = 0 is the apex),



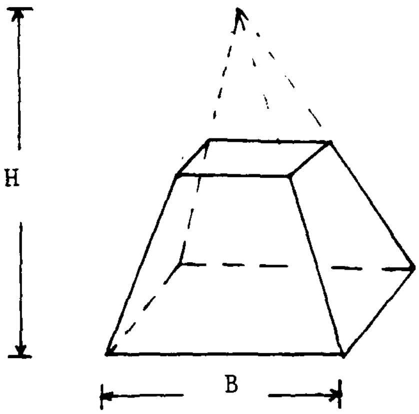
$$y = f(z) = (B/2H)z$$

this reduces to

$$V = \int_0^H (2y)^2 dz$$
$$= \int_0^H (B^2/H^2)z^2 dz$$

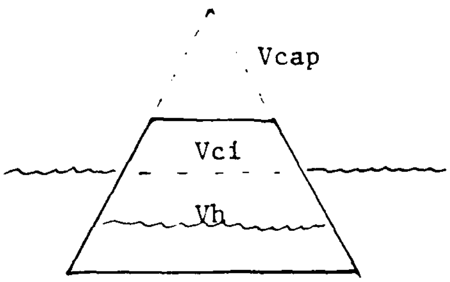
Therefore,

$$V = (B^2/3H^2)z^3$$



At z = H, the volume of the pyramid is one-third that of a column of the same height having the same base. This formula can be used in the same way as the corresponding formulas for the cone were used in the apex up case.

The subtraction method is illustrated below.



$$Vh = Vt = Vpyramid - Vci - Vcap = Vpyr - (Vci + Vcap)$$
$$= (B^2/3H^2)[(zc + hi + h)^3 - (hi + zc)^3]$$

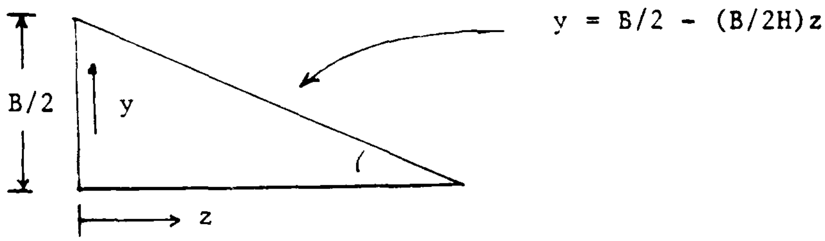
The expression in square brackets is the same as in the cone, apex up case. Therefore by analogy,

$$Vh = Vt = (B^2/3H^2)(h)[h^2 + 3h(hi + zc) + 3(hi + zc)^2]$$

Very little simplification is possible because the terms are all about the same size.

7. Pyramid (Apex Down)

Here again, the volume formula is rewritten with z = 0 at the base, and the volume formula rederived.



$$V = \int_0^z A(z)dz$$
$$= \int_0^z (2y)^2 dz$$
$$= \int_0^z (B - Bz/H)^2 dz$$
$$= \int_0^z (B^2 - 2B^2z/H + B^2z^2/H^2)dz$$
$$V = B^2z - B^2z^2/H + (B^2/3H^2)z^3$$
$$V = (B^2/3H^2)(z)[3H^2 - 3Hz + z^2]$$

Inverted, in the water, the cap can be neglected, when a formula which starts from the base is available (see above).

Then

$$Vh = Vt = V(h + hi) - Vhi$$

$$Vt = (B^2/3H^2)(h + hi)[3H^2 - 3H(h + hi) + (h + hi)^2]$$
$$- (B^2/3H^2)(hi)[3H^2 - 3Hhi + hi^2]$$

Multiplying terms and cancelling,

$$Vt = (B^2/3H^2)[3H^2h - 3Hh^2 - 3Hhih + h^3 + 2h^2hi$$
$$+ hi^2h + 3H^2hi - 3Hhi^2 - 3Hhih + h^2hi$$
$$+ 2hhi^2 + hi^3 - 3H^2hi + 3Hhi^2 - hi^3]$$
$$= (B^2/3H^2)[3H^2h - 3Hh^2 - 6Hhih + h^3 + 3h^2hi + 3hi^2h]$$
$$= (B^2/3H^2)h[3H^2 - 3Hh - 6Hhi + h^2 + 3hhi + 3hi^2]$$

or,

$$Vt = (B^2/3H^2)(h)[3(H - hi)^2 - 3h(H - hi) + h^2]$$

which is quite similar to the corresponding equation for the cone. Further simplification is not justified.

Predictions of pressure - volume relationships for the two pyramid orientations, apex up and apex down, at various free - volumes (immersion depths) are given in Figs. 15 and 16, following.

FIG. 15. PRESSURE–VOLUME CURVES FOR PYRAMID, APEX UP

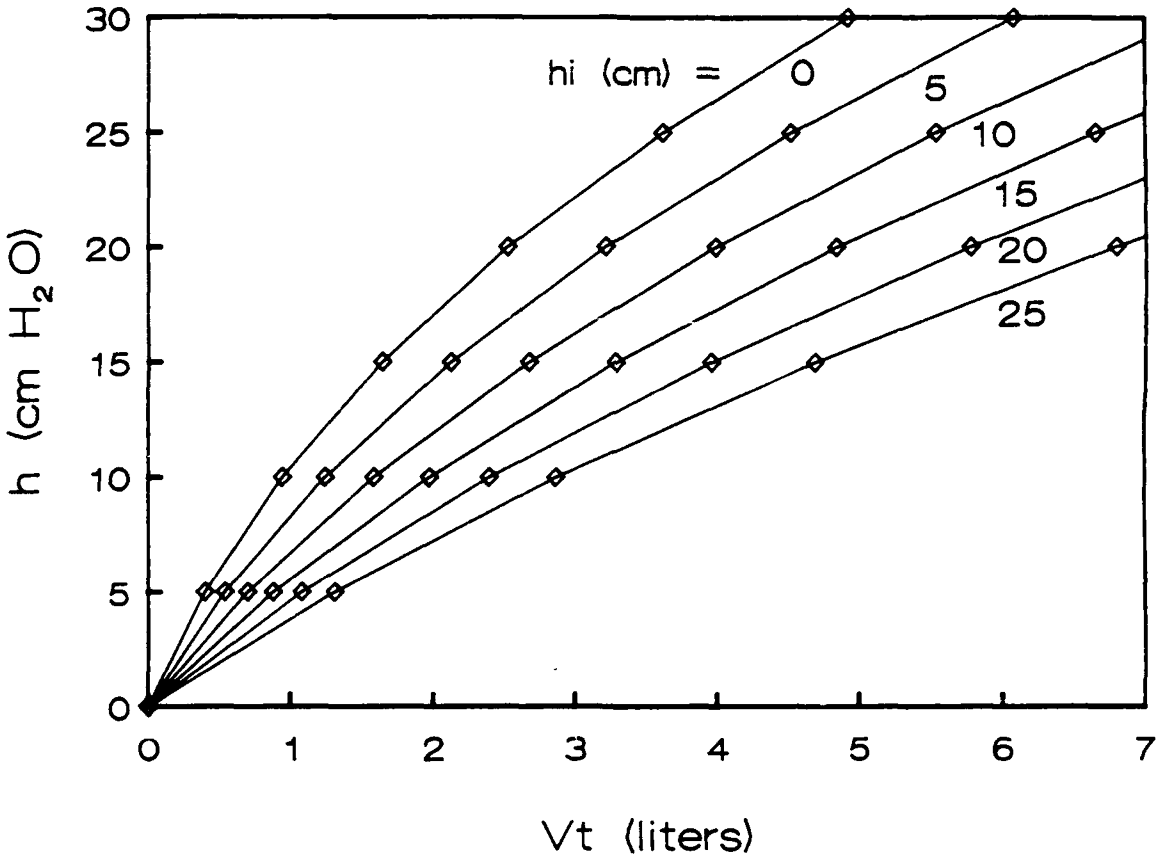
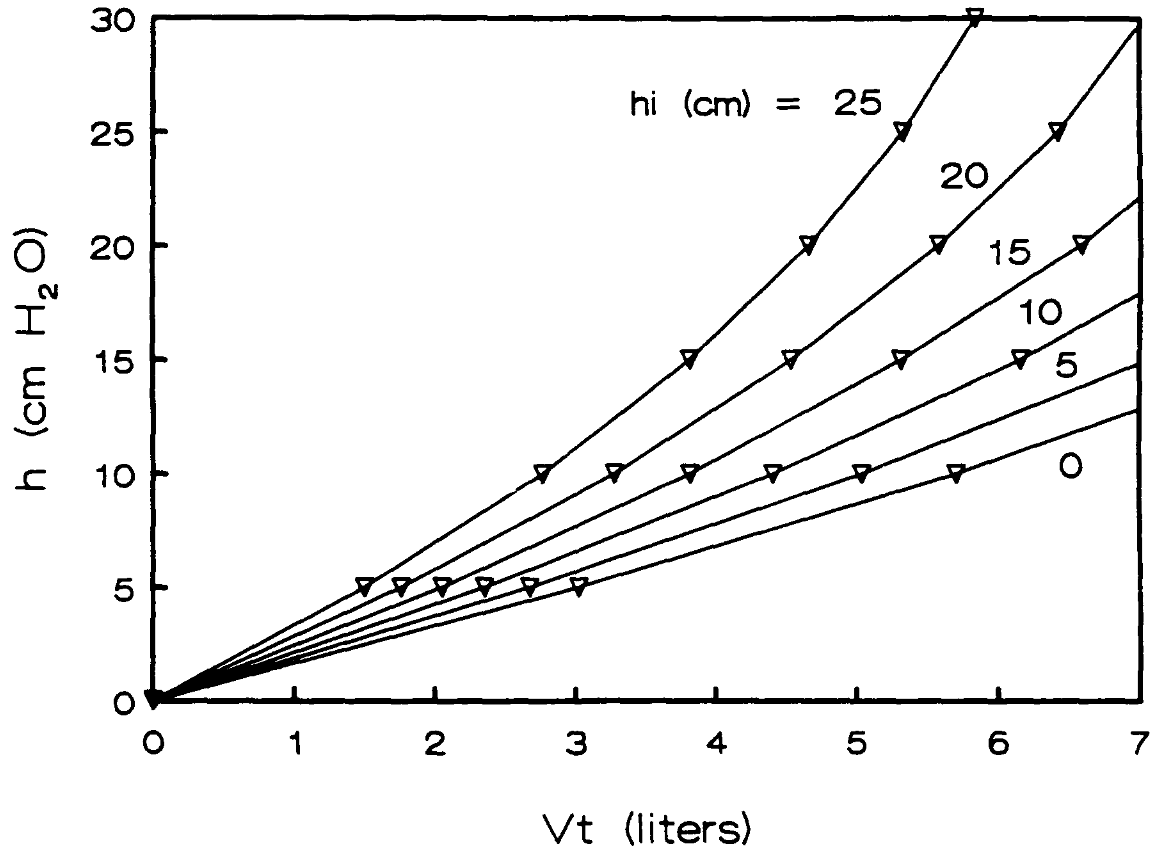


FIG. 16. PRESSURE-VOLUME CURVES FOR PYRAMID, APEX DOWN



APPENDIX C: BALLOON/BREATHING BAG ELASTANCE FORMULAS

Balloon Pressure-Volume Curve

The pressure-volume relationship for a spherical rubber balloon in air can be developed from a force balance in which the net pressure force on the inside (or pressure difference between inside and outside the balloon) is balanced by the elastic tensile force in the balloon material. If the balloon were in water, an additional pressure due to hydrostatic head would be present. This would also change the shape of the balloon from spherical and complicate the analysis considerably. Thus, the following is for a balloon in air, which is another geometry that can be used to generate elastic loads.

Since pressure and tension are both force/area terms that act over different areas, the force balance equation can be written

$$(\Delta P)\ (\pi r^2) = (T)\ (2\pi r\delta) \tag{21}$$

where $\Delta P = h$ = pressure difference from atmospheric, r is the sphere radius which changes as the balloon enlarges, δ is the thickness of the material of the balloon skin, and T is the tensile stress. The above equation results from multiplying the tension around a hemispherical slice of the balloon (visualize an orange cut in half) by the appropriate sectional area (the slice through the orange skin) and setting this equal to the appropriate, oppositely directed resultant force generated by the pressure difference acting on a projection of the hemispherical shell.

For a rubbery elastic material the best measure of strain is the natural logarithm of the ratio of radii (7). Thus,

$$T = E' \varepsilon = E' \ln(r/r0) \tag{22}$$

where E' is the elastic modulus (Young's modulus) and r0 is the point at which the balloon is first fully shaped (spherical) when being expanded from rest or from a flaccid or collapsed state. The corresponding volume, V0, is

$$V0 = (4/3)\pi\ r0^3 \tag{23}$$

The elastic modulus is normally a constant, but depending on the material, it may increase or decrease with expansion of the balloon.

Rearranging the force balance equation yields,

$$\Delta P = h = T(2\delta/r) \tag{24}$$

but δ, the thickness, is a function of r, also; the thickness is less at larger r. If the material is assumed incompressible (a good assumption in fact), then the original volume of rubber material, V*, is conserved.

$$V^* = 4\pi r^2 \delta \tag{25}$$

at all stages from V0 or r0. Therefore,

$$\delta = V^*/(4\pi r^2) \tag{26}$$

Substituting in gives

$$h = (2V^*/4\pi r^3)[E'\ \ln\ (r/r0)] \tag{27}$$

which can also be written in shorthand form as

$$h = C1 \ln (Vb/V0)^{1/3}/Vb \tag{28}$$

where Vb is the balloon volume, and C1 and V0 are constants. This is a complex expression that cannot be inverted to give a simple form, Vb = f(h), for substitution into the general analysis. It does give a nonlinear elastance relationship for the balloon.

That equation (28) has a maximum can be demonstrated by taking the derivative with respect to r or V, setting it equal to zero and solving the resulting equation. For example,

$$dh/dr = 0 = d[C1 \ln(Vb/V0)^{1/3}/Vb]/dr$$

$$0 = d[C1 \ln(r/r0)/(4\pi r^3/3)]dr$$

$$0 = 1/r^4 - 3 \ln(r/r0)/r^4$$

$$\ln(r/r0) = 1/3$$

$$r = r0e^{1/3}$$

$$rpmax = 1.3956\ r0$$

Or in terms of volume

$$(V/V0)^{1/3} = e^{1/3} = r/r0$$

$$Vpmax = e\ (V0) = 2.7183\ (V0) \tag{29}$$

where Vpmax is the volume at maximum pressure in a balloon with spherical geometry and constant modulus of elasticity made from incompressible rubber.

That equation (28) also has an inflection point can be demonstrated by taking the second derivative and setting it equal to zero.

$$d(dh/dV)/dV = 0 = d\{d[(C1/Vb)\ \ln(Vb/V0)^{1/3}]\}/dV$$

$$0 = d\{(-C1/Vb^2)\ \ln(Vb/V0)^{1/3} + (C1/Vb)[d\ \ln\ (Vb/V0)^{1/3}/dV]\}/dV$$

$$= d[(-C1/Vb^2)\ \ln(Vb/V0)^{1/3} + (C1/Vb)(Vb/V0)^{-1/3}(1/3)(Vb/V0)^{-2/3}/V0]/dV$$

$$= d[(-C1/Vb^2)\ \ln(Vb/V0)^{1/3} + (C1/Vb)(1/3Vb)]/dV$$

$$= d[(-C1/Vb^2)\ \ln(Vb/V0)^{1/3} + C1/3Vb^2]/dV$$

$$= 2C1\ \ln(Vb/V0)^{1/3}/Vb^3 - (C1/Vb^2)\ d\ \ln\ (Vb/V0)^{1/3}/dV - 2C1/3Vb^3$$

$$= 2C1\ \ln(Vb/V0)^{1/3}/Vb^3 - (C1/Vb^2)(1/3Vb) - 2C1/3Vb^3$$

$$= 2C1\ \ln(Vb/V0)^{1/3}/Vb^3 - C1/Vb^3$$

$$= 2\ \ln\ (Vb/V0)^{1/3} - 1$$

$$\ln\ (Vb/V0)^{1/3} = 1/2$$

$$(Vb/V0)^{1/3} = e^{1/2}$$

$$Vb/V0 = e^{3/2} = 4.4817 \qquad (30)$$

Plotting equation (28) will show both the maximum and the inflection point.

General Analysis

An elastic system with the balloon is generally similar to the water column case, only the new volume is generated by an expanding balloon instead of a moving water column. Thus,

$$Pfinal/Patm = (Vsi + Vx)/(Vsi + Vx + Vb - Vt) \qquad (31)$$

Equation (31) is not valid for Vb less than V0; it only applies to inflated balloons in air, not limp or flaccid balloons. Rearranging as before gives,

$$h = 1033[(Vt - Vb)/(Vsi + Vx + Vb - Vt)]$$

The above equation as it stands is not satisfactory, because both Vt and Vb are variable and almost equal. An experimental verification of this equation was attempted, but was unsatisfactory owing to the difficulty in getting the extreme precision needed in the volume measurements. Therefore, the pressure-volume relationship for the elastic balloon was zoned and linearized. This procedure gave a quadratic equation which could be solved as before. The simplification step gave the (now linearized) balloon pressure - volume relationship.

Zone I is the flaccid zone up to V0, where h is zero. In zone II, the steeply increasing curve of the balloon pressure - volume relationship is approximated by

$$h = m\,Vb - q$$

where m is the slope and q the y-intercept of the straight line approximation. This can be written,

$$Vb = h/m + q/m$$

and substituting in to the general equation,

$$h = 1033[(Vt - h/m - q/m)/(Vsi + Vx - Vt + h/m + q/m)]$$

Rearranging and solving for h

$$h = 1033(m\ Vt - h - q)/[m(Vsi + Vx - Vt) + h + q]$$

Cross-multiplying and rearranging gives the quadratic form

$$h^2 + [m(Vsi + Vx - Vt) + q + 1033)]h - 1033(m\ Vt - q) = 0$$

from which

$$h = -[m(Vsi + Vx - Vt) + q + 1033]/2 + \\ + \mathrm{SQRT}\ [(b/2a)^2 + 1033(m\ Vt - q)] \qquad (32)$$

Simplifying terms from the quadratic expression by taking 1033 larger than all other pressures gives

$$h = m\ Vt - q$$

which is the zone II equation.

The zone III analysis follows in the same manner. The zone III relationship and simplified form of the general relationship is

$$h = q - mVt$$

and the general expression is derived from

$$h = 1033(mVt + h - q)/[m(Vsi + Vx - Vt) - h + q]$$

from which

$$h = -[1033 - q - m(Vsi + Vx - Vt)]/2 + \\ + \mathrm{SQRT}\ [(b/2a)^2 - 1033(mVt - q)] \qquad (33)$$

APPENDIX D: BALLOON/BREATHING BAG DATA

1. Punch-Ball Balloon Data

Vt from syringe, Vb from plethysmograph syringe, h from water manometer.

Vt, liters	h, cm H20	Vb, liters
0	0	0
0.5	13.2	0.45
1.0	36.2	.84
1.5	34.5	1.36
2.0	32.5	1.88
2.5	30.7	2.40
3.0	30.0	2.73
.1	0	0.1
.2	0	.2
.3	0	.31
.4	0	.405
.5	5.1	.50
repeat average	9.46	.47
.6	15.0	.565
.7	22.7	.635
.8	27.3	.725
.9	30.5	.82
1.0	32.6	.91
repeat average	34.15	.87
1.1	33.4	1.05
1.2	33.5	1.11
1.3	33.4	1.21
1.4	33.4	1.31
1.5	33.1	1.41

2. Curve Fit Predictions Using Simplified Zone Equations and Comparison with Experimental Data

	Vt, liters	hexp, cm H_2O	hpred, cm H_2O
zone II			
	0.5	9.46	10.0
	.6	15.0	15.1
	.7	22.7	20.1
	.8	27.3	25.1
	.9	30.5	30.2
	1.0	34.2	35.3
zone III			
	1.1	33.4	34.6
	1.2	33.5	34.3
	1.3	33.4	33.0
	1.4	33.4	33.6
	1.5	33.1	33.3
	2.0	32.5	32.6
	2.5	30.7	30.0
	3.0	29.7	28.3

3. Balloon Data and Curve Fit with Vb

Vt	hexp	Vb	htheory
.5	9.46	.467	9.46
.6	15.0	.565	20.7
.7	22.7	.635	25.4
.8	27.3	.725	29.3
.9	30.5	.820	31.6
1.0	34.2	.870	32.4
1.1	33.4	1.05	33.7
1.2	33.5	1.11	33.8
1.3	33.4	1.21	33.7
1.4	33.4	1.31	33.4
1.5	33.1	1.41	33.1
2.0	32.5	1.86	30.8
2.5	30.7	2.38	28.0
3.0	29.7	2.73	26.3

best fit constants C1 = 114.6, V0 = .416

Equation, $h = C1 \ln(Vb/V0)^{1/3}/Vb$, is solved simultaneously at Vt's of 0.5 and 1.5. Thus, comparison is exact in these two cases.

4. Red Weather Balloon

Vt, liters	h, cm H20
0	0
21	0
24	0.6
27	2.1
30	3.2
33	4.0
36	4.3
39	4.6
42	5.0
45	5.0
"	"
102	5.0
105	5.2
"	"
135	5.2

balloon broke at 135

5. 5-liter Breathing Bag

Vt	h
0	0
3.5	0
4.0	0.3
4.5	1.0
5.0	6.0
5.5	12.0
6.0	18.0

6. 15-liter Breathing Bag

Vt	h
0	0
11	0
12	0.5
13	1.8
13.5	3.6
14	7.0
14.5	10.0
15	15.0

APPENDIX E: ELASTIC LOADING DATA FROM THE MK 15 TRIALS

1. Run 1

IN AIR

Vt, liters	h(up)	h(side)	h(down)
0	0	0	0
0.5		0.5	0.4
1.0		0.5	0.6
1.5		0.7	2.8
1.7		-	3.5*
2.0		1.4	
2.2		1.7	
2.5		3.8	
3.0		4.6*	
4.0	4.8*		

h(up) is with the rig horizontal, face up
h(side) is with the rig on its side
h(down) is with the rig horizontal, face down
* pop-off valve opened.

1. Run 1 (cont.)

IN WATER

Vt	h(down)	h(vert)	h(angle)	h(up)
0	0	0	0	0
0.1	5.5	4.4	-	-
0.2	8.8	7.3	8.7	6.1
0.4	13.5	9.7	13.0	9.3
0.5	14.2	11.1	14.0	-
0.6	14.9	12.3	14.4	10.9
0.8	15.7	-	15.8	-
1.0	16.0	13.8	16.4	-
1.2	16.3	14.9	16.9	11.2
1.4	16.8	15.7	17.3	12.2
1.5	17.0	16.0	17.5	-
1.6	17.3	16.3	19.7	12.7
1.8	17.7	16.9	18.2	13.2
2.0	18.1	17.8	18.5	13.6
2.2	18.7	18.1	-	13.4
2.4	19.1	19.2	-	14.4
2.5	19.3	19.8	20.4	-
2.6	19.4	20.4	-	15.2
2.8	19.7	21.4	-	15.7
3.0	20.0	22.6	22.5	16.0
3.2	20.4	23.0	-	16.2
3.4	20.6	23.6	-	16.2
3.5	20.7	23.9	23.8	-
3.6	21.0	24.0	-	16.2
3.8	21.5	24.8	-	16.2
4.0	22.1	25.5	25.3	16.2
4.5	23.4	26.9	26.0	15.4
5.0	26.1	28.6	28.4	18.7
5.2	-	29.6*	-	19.2*
5.5	28.1*		30.4*	

* bubbles due to pop-off valve opening.

2. Run 2

IN AIR

Vt	h(down)	h(side)	h(up)
0	0	0	0
0.2	0	0.1	0.6
0.5	0.2	.2	.85
0.8	.4	.3	-
1.0	1.3	.3	.85
1.3	1.9	-	-
1.5	3.8	.3	.9
1.7	4.4	-	-
2.0	4.9*	.8	.9
2.5		1.2	1.05
3.0		3.6	1.4
3.5		5.2*	1.8
4.0			3.0
4.3			5.5*

IN WATER

Vt	h(down)	h(vert)	h(angle)	h(up)
0.2	8.6	6.5	8.4	7.3
0.5	13.9	11.0	14.7	10.7
0.8	15.2	11.3	16.0	11.0
1.0	15.3	13.5	16.5	12.3
1.5	16.2	16.0	17.7	13.1
2.0	17.6	18.1	18.4	13.6
2.5	18.3	20.2	19.7	14.5
3.0	18.7	22.9	23.0	15.7
3.5	19.7	24.1	24.2	16.7
4.0	20.4	25.6	25.7	17.1
4.5	21.8	26.7	26.8	16.7 ?
5.0	24.1	27.9	29.0	20.7
5.2	25.5	29.9	30.8*	19.6*
5.5	26.8*	30.1*		

NOTE: The face is the side of the rig next to the diver's back.
* pop-off valve opening
? questionable value

3. Mk 15 Miscellaneous

a) Approximate dimensions of canister
diameter = 12 inches
depth = 6 inches

b) Volume estimate

Obtaining an estimate of Vsumi (the volume excluding bag volume) for UBA by using a syringe with pressure measurement and Boyle's law was difficult to do, because of leaks. By geometry measurements, the canister alone was estimated to be 11.1 liters and the hoses 1.74 liters, which gives a total volume of 12.8 liters. Actual volume is likely to be slightly higher because of other tubing, etc., not accounted for here.

APPENDIX F: ELASTANCE DATA FOR PYRAMID

		APEX UP	APEX DOWN	
	Vt, liters	h, cm H20	h, cm H20 pull	0.5
-2.4		-1.2		
	1.0	-5.3	-2.4	
	1.5	-8.8	-3.6	
	2.0	-13.0	-4.8	
	2.5	-18.7	-6.0	
	3.0	-	-7.1	
push	0.5	2.3	1.1	
	1.0	4.6	2.5	
	1.5	6.5	3.8	
	2.0	8.3	5.5	
	2.5	10.2	7.0	
	3.0	11.9	8.5	
	*3.5	13.2	10.2*	
	4.0	14.8	12.1	
	4.5	16.2	14.0	
	5.0	17.6	15.9	
	5.5	19.0	18.2	
	6.0	20.3	20.7	
	6.5	21.5	23.3	
	7.0	22.7	26.4	

* 7-liter syringe for remaining data. Data up to this point used 3-liter syringe. Pyramid dimensions: top side = 8.2 cm, bottom side (B) = 25.3 cm, length = 59.0 cm. Calculated dimensions: H = 87.3 cm, zc = 28.3 cm. Free space zone hi = 20 cm in both cases. Tubing volume (30 inches 1.5 in tubing), Vx = 0.88 liters.